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Explain your work
1. **(25 points)** Show that the systems of differential equations corresponding to any two mass-spring oscillators

\[ x'' + bx' + kx = 0, \quad x'' + \beta x' + \kappa x = 0, \]

with \( b, k, \beta, \kappa > 0 \), are topologically conjugate.

**Solution:** The matrices of the corresponding systems are

\[ A = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ -\kappa & -\beta \end{bmatrix}. \]

The traces of \( A \) and \( B \) are \(-b < 0\) and \(-\beta < 0\), respectively, \( \det A = k > 0 \) and \( \det B = \kappa > 0 \). This implies that both systems lie in the second (i.e., upper left) quadrant of the trace-determinant plane and are therefore hyperbolic (spiral or real) sinks. By the classification theorem for hyperbolic planar linear systems, they are topologically (though not necessarily smoothly) conjugate.
2. (25 points) Show that the equations

\[ x' = -x \quad \text{and} \quad y' = y + y^5 \]

are not topologically conjugate.

(Hint: You do not need to solve either equation.)

Solution: Both systems have a unique equilibrium at the origin. For \( x' = -x \), the equilibrium is a sink, whereas for \( y' = y + y^5 \), it is a source, since

\[ y + y^5 = y(1 + y^4) \begin{cases} > 0 & \text{if } y > 0 \\ < 0 & \text{if } y < 0. \end{cases} \]

Since their solutions do not share the same fate, the equations are not topologically conjugate.

More rigorously, suppose that \( h \) is a conjugacy; then \( h(0) = 0 \), since equilibria have to be mapped to equilibria. If \( x(t) \) is any solution to the first equation, then \( y(t) = h(x(t)) \) is a solution to the second equation and

\[ \lim_{t \to \infty} y(t) = \lim_{t \to \infty} h(x(t)) = h(\lim_{t \to \infty} x(t)) = h(0) = 0, \]

which is impossible, since all solutions to the second equation are repelled from the origin.
3. (25 points) Let $A$ be a $2 \times 2$ matrix.

(a) Given that

$$e^{tA} = \frac{1}{2} \begin{bmatrix} e^{2t} + e^{-t} & e^{2t} - e^{-t} \\ e^{2t} - e^{-t} & e^{2t} + e^{-t} \end{bmatrix},$$

for all $t \in \mathbb{R}$, find $A$.

(b) For which initial conditions does the corresponding solution of $X' = AX$ converge to the origin?

Solution: (a) We have:

$$A = \frac{d}{dt} \bigg|_{t=0} e^{tA} = \frac{1}{2} \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}. $$

(b) The flow of $X' = AX$ is given by

$$\phi_t(X) = e^{tA}X$$

$$= \frac{1}{2} \begin{bmatrix} e^{2t} + e^{-t} & e^{2t} - e^{-t} \\ e^{2t} - e^{-t} & e^{2t} + e^{-t} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} e^{2t}(x+y) + e^{-t}(x-y) \\ e^{2t}(x+y) - e^{-t}(x-y) \end{bmatrix}. $$

Therefore, $\phi_t(x, y) \to (0, 0)$, as $t \to \infty$, if and only if the $e^{2t}$-terms are zero, i.e., $x+y = 0$. Alternatively, the eigenvalues of $A$ are $-1$ and $2$, and the eigenspace corresponding to $-1$ is the line $y = -x$. 
4. (25 + 10 points) The flow of the system of differential equations

\[ \begin{align*}
    x' &= f(x, y) \\
    y' &= g(x, y)
\end{align*} \]

is given by

\[ \phi_t(x, y) = \left( (x + \frac{1}{5}y^3)e^{2t} - \frac{1}{5}y^3e^{-3t}, ye^{-t} \right). \]

(a) Determine the system, i.e., compute \( f(x, y) \) and \( g(x, y) \).

(b) Find the equilibria.

(c) Are there any periodic solutions?

(d) (extra credit: 10 points) Find the stable and unstable manifolds.

Solution: (a) By definition of the flow,

\[
(f(x, y), g(x, y)) = \frac{d}{dt} \bigg|_{t=0} \phi_t(x, y) = (2x + y^3, -y).
\]

Therefore, the system is

\[ \begin{align*}
    x' &= 2x + y^3 \\
    y' &= -y.
\end{align*} \]

(b) The only equilibrium is the origin.

(c) There are no periodic solutions, since the functions \( e^{at} \) are not periodic.

(d) The stable manifold is the set of all \((x, y) \in \mathbb{R}^2 \) such that \( \phi_t(x, y) \to (0, 0) \), as \( t \to \infty \). Therefore,

\[ W^s(0) = \{(x, y) : x + \frac{1}{5}y^3 = 0\}. \]

The unstable manifold is the set of points \((x, y) \) such that \( \phi_t(x, y) \to (0, 0) \), as \( t \to -\infty \). Therefore,

\[ W^u(0) = \{(x, y) : y = 0\}. \]