San José State University

Math 213, Fall 2007

Midterm Solutions

October 24, 2007

Due on October 29, 2007

Honor system is in effect.

Name: Granwyth Hultaberi

	Score
1	25
2	25
3	25
4	25
XC	25
Total	125

All manifolds are assumed to be of class C^∞.

1. (25 points) Let $f : M \to M$ be a diffeomorphism. If $f(p) = p$, p is called a fixed point of f. A fixed point p is called a Lefshetz fixed point if 1 is **not** an eigenvalue of $T_p f : T_p M \to T_p M$. If all fixed points of f are Lefshetz, f is called a Lefshetz map.

(a) Show that a Lefshetz fixed point is isolated, i.e., it has a neighborhood U such that f has no other fixed points in U.

(b) If M is compact and f is a Lefshetz map, show that f has finitely many fixed points.

Proof: (a) If p is a Lefshetz fixed point of f, then for any coordinate neighborhood (U, φ) of p, $\varphi(p)$ is a Lefshetz fixed point of $\tilde{f} = \varphi^{-1} \circ f \circ \varphi$. This is because

$$\tilde{f}_* = (\varphi_*)^{-1} f_* \varphi_*,$$

so $\tilde{f}_*$ and f_* have the same eigenvalues.

Therefore, it is enough to prove the statement of (a) for $M = \mathbb{R}^n$. So let p is a Lefshetz fixed point of a diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ and consider the equation $f(x) = x$ in a neighborhood of p. We can rewrite it as

$$g(x) = \mathbf{0},$$

where $g(x) = f(x) - x$. Since

$$Dg(p) = Df(p) - I,$$

and 1 is not an eigenvalue of $Df(p)$, $Dg(p)$ is non-singular. By the Inverse Function Theorem, g is a local diffeomorphism near p. That is, there exists a neighborhood U of p and a neighborhood V of $\mathbf{0} = g(p)$ such that $g : U \to V$ is a diffeomorphism. In particular, the equation $g(x) = \mathbf{0}$ has a unique solution p in U. This means that the only fixed point of f in U is p.

(b) Suppose that (p_k) is a sequence of fixed points of f. By compactness of M, it has a convergent subsequence, $p_{k_i} \to p$, as $i \to \infty$. Since $f(p) = f(\lim p_{k_i}) = \lim f(p_{k_i}) = \lim p_{k_i} = p$, p is a fixed point of f. But this is impossible, since all fixed points of f are Lefshetz, hence isolated by (a).

2. (25 points) Recall that $G(k, n)$ denotes the Grassmann manifold of k-planes in $\mathbb{R}^n$.

(a) Show that $O(n)$ acts transitively on $G(k, n)$.

(b) Find the isotropy subgroup of the k-plane

$$E = \text{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_k\},$$

where $\mathbf{e}_i$ denotes the i^{th} vector in the standard orthonormal basis for $\mathbb{R}^n$.

Solution: (a) We will show that the orbit of the k-plane E defined in (b) is all of $G(k, n)$. Let $F \in G(k, n)$ be arbitrary. Pick an orthonormal basis $\{f_1, \ldots, f_k\}$ of F and using Gramm-Schmidt, extend it to an orthonormal basis $\{f_1, \ldots, f_n\}$ of $\mathbb{R}^n$. Let A be the matrix whose columns are $f_1, \ldots, f_n$. By construction, $A \in O(n)$ and $Ae_i = f_i$, i.e., $AE = F$. This proves that the orbit of E is $G(k, n)$, so the action is transitive.

(b) Suppose A is in the isotropy group G of E, i.e., $AE = E$. Since A is an orthogonal matrix, $AE^{\perp} = E^{\perp}$, so A is a block matrix of the form

$$\begin{bmatrix} P & \mathbf{0} \\ \mathbf{0} & Q \end{bmatrix},$$

where $P \in O(E) \cong O(k)$ and $Q \in O(E^{\perp}) \cong O(n-k)$. Therefore, G is isomorphic to $O(k) \times O(n-k)$.

3. (25 points) Let X be a C^∞ vector field on a manifold M. Assume that X is non-vanishing, i.e., $X(p) \neq 0$, for all $p \in M$ and let $f : M \to \mathbb{R}$ be an arbitrary C^∞ function. Show that the equation

$$Xu = f$$

is locally solvable. That is, for every $p \in M$ there exists a neighborhood U of p and $u \in C^\infty(U)$ such that $Xu = f$.

Proof: By the Flow Box Theorem, X locally looks like the vector field $\partial/\partial x_1$ and the equation $\partial u/\partial x_1 = f$ is clearly solvable. More precisely, for every $p \in M$ there exists a coordinate neighborhood (U, φ) such that $\varphi(p) = \mathbf{0}$, $\varphi(U)$ is a cube centered at $\mathbf{0}$, and

$$\varphi_*(X) = \frac{\partial}{\partial x_1}.$$

Let $f \in C^\infty(M)$ be arbitrary. In this coordinate neighborhood, the PDE $Xu = f$ translates into

$$\frac{\partial u}{\partial x_1} = \tilde{f}, \tag{1}$$

where $\tilde{f} = f \circ \varphi^{-1}$. For $x = (x_1, \ldots, x_n) \in \varphi(U)$ define

$$\tilde{u}(x) = \int_0^{x_1} \tilde{f}(t, x_2, \ldots, x_n)\, dt.$$

Then $\tilde{u}$ is a solution to (1) on $\varphi(U)$. Let $u = \tilde{u} \circ \varphi$. Clearly, $u \in C^\infty(U)$. Recalling that $\varphi_*(X)\tilde{u} = X(\tilde{u} \circ \varphi) = (Xu) \circ \varphi^{-1}$, we obtain

$$\begin{aligned}(Xu) \circ \varphi^{-1} &= \varphi_*(X)\tilde{u} \\ &= \frac{\partial \tilde{u}}{\partial x_1} \\ &= \tilde{f} \\ &= f \circ \varphi^{-1}.\end{aligned}$$

Since $\varphi : U \to \varphi(U)$ is a diffeomorphism, it follows that $Xu = f$ on U.

4. (25 points) Prove the following statements.

(a) The map $\mathscr{P} : G(k,n) \to G(n-k,n)$ defined by

$$\mathscr{P}(E) = E^{\perp}$$

is a diffeomorphism.

(b) The Grassmann manifold $G(2,3)$ and the projective 2-space $P^2(\mathbb{R})$ are diffeomorphic.

Hint: Let E be a k-plane in $\mathbb{R}^n$. Denote its orthogonal complement by $E^{\perp}$ and identify $\mathbb{R}^n$ with $E \times E^{\perp}$. There exists a neighborhood $\mathscr{U}$ of E in $G(k,n)$ such that every $F \in \mathscr{U}$ is the graph of a unique linear map $L : E \to E^{\perp}$. Define $\varphi : \mathscr{U} \to \mathscr{L}(E, E^{\perp})$ by $\varphi(F) = L$, where $\mathscr{L}(E, E^{\perp})$ denotes the space of linear maps $E \to E^{\perp}$. Then $\{(\mathscr{U}, \varphi)\}$ is a smooth structure on $G(k,n)$, equivalent to the one defined in Boothby. Feel free to use it.

Proof: (a) Since $(E^{\perp})^{\perp} = E$, $\mathscr{P}$ is clearly 1–1 and onto, so it suffices to show that it is smooth.

Let $E \in G(k,n)$ be fixed. Identify $\mathbb{R}^n$ with $E \times E^{\perp}$, that is, identify E with $\mathbb{R}^k \times \{\mathbf{0}_{n-k}\}$ and $E^{\perp}$ with $\{\mathbf{0}_k\} \times \mathbb{R}^{n-k}$. Every $w \in \mathbb{R}^n$ can be written in a unique way as the sum $u + v$, where $u \in E$ and $v \in E^{\perp}$.

Let $(\mathscr{U}, \varphi)$ be a coordinate neighborhood of E in $G(k,n)$ as defined in the hint with $\varphi : \mathscr{U} \to \mathscr{L}(E, E^{\perp})$. Let $\mathscr{V} = \mathscr{P}(\mathscr{U})$ be the corresponding coordinate neighborhood of $E^{\perp}$ in $G(n-k,n)$ and $\psi : \mathscr{V} \to \mathscr{L}(E^{\perp}, E)$ the corresponding coordinate map.

Let $F \in \mathscr{U}$ be arbitrary and set $L = \varphi(F)$. This means that F is the graph of L, i.e.,

$$F = \{u + Lu : u \in E\}.$$

How do we express $F^{\perp}$ as the graph of some $K \in \mathscr{L}(E^{\perp}, E)$? We therefore want

$$F^{\perp} = \{Kv + v : v \in E^{\perp}\}.$$

We are therefore looking for a linear map $K : E^{\perp} \to E$ such that

$$(u + Lu) \cdot (Kv + v) = 0,$$

for all $u \in E$ and $v \in E^{\perp}$. Using $u \cdot vKv = Lu \cdot Kv = 0$, this simplifies to

$$u \cdot Kv = -Lu \cdot v.$$

Since $Lu \cdot v = u \cdot L^* v$, where L^* is the adjoint (i.e., transpose) of L, it follows that $K = -L^*$. If we choose a basis of E and $E^{\perp}$, e.g., the standard basis, the matrix of K is just the negative transpose of the matrix of L. Since the map

$$\tilde{\mathscr{P}}(L) = -L^*$$

is linear, it is smooth. Therefore, $\mathscr{P}$ is smooth, hence a diffeomorphism.

(b) By definition $G(1,3) = P^2(\mathbb{R})$ and by (a), $G(2,3) \approx G(1,3)$.

Extra credit (25 points) Let X and Y be C^∞ vector fields on a compact manifold M and assume

$$[X, Y] = Z, \quad [X, Z] = 0, \quad [Y, Z] = 0.$$

Denote the (time-t maps of the) flows of X, Y, Z by X^t, Y^t, Z^t, respectively.

(a) Show that

$$(X^t)_*(Y) = Y - tZ,$$

for all $t \in \mathbb{R}$.

(b) Let $t \in \mathbb{R}$ and $p \in M$ be fixed and define a curve γ by

$$\gamma(s) = (Y^{-s} \circ X^{-t} \circ Y^s \circ X^t)(p).$$

Show that

$$\dot{\gamma}(s) = tZ(\gamma(s)),$$

for all $s \in \mathbb{R}$.

(c) Show that for all $t \in \mathbb{R}$ and $p \in M$.

$$(Y^{-t} \circ X^{-t} \circ Y^t \circ X^t)(p) = Z^{t^2}(p).$$

Proof: Recall that the assumptions $[X, Z] = 0$ and $[Y, Z] = 0$ imply the following

Fact: $X^t_*(Z) = Z$ and $Y^t_*(Z) = Z$.

(a) Let $p \in M$ be fixed and denote by $V(t)$ the value of $(X^t)_*(Y)$ at p. That is,

$$V(t) = T_{X^{-t}p} X^t (Y(X^{-t}p)).$$

Then $t \mapsto V(t)$ is a C^∞ curve in T_pM satisfying $V(0) = Y(p)$. Recall that

$$\left.\frac{d}{dt}\right|_0 X_*^{-t}(Y) = L_X Y.$$

If we differentiate not at zero but at some t, then

$$\begin{aligned}
\frac{d}{dt} X_*^{-t}(Y) &= \left.\frac{d}{ds}\right|_{s=0} X_*^{-(t+s)}(Y) \\
&= \left.\frac{d}{ds}\right|_{s=0} X_*^{-t} X_*^{-s}(Y) \\
&= X_*^{-t}\left(\left.\frac{d}{ds}\right|_{s=0} X_*^{-s}(Y)\right) \\
&= X_*^{-t}(L_X Y),
\end{aligned}$$

Therefore,

$$\dot{V}(t) = -X_*^{-t}(L_X Y) = -X_*^{-t}(Z) = -Z,$$

evaluated at p, of course. By the Fundamental Theorem of Calculus,

$$V(t) = V(0) + \int_0^t (-Z(p))\, ds = Y(p) - tZ(p).$$

(b) By the Chain Rule, we have:

$$\begin{aligned}
\dot{\gamma}(s) &= -Y + Y_*^{-s} X_*^{-t}(Y) \\
&\overset{(a)}{=} -Y + Y_*^{-s}(Y + tZ) \\
&= -Y + Y_*^{-s}(Y) + tY_*^{-s}(Z) \\
&= -Y + Y + tZ \\
&= tZ. \qquad (*)
\end{aligned}$$

Here we used the fact that the vector field is invariant under its flow and $Y_*^{-s}(Z) = Z$. Since $\dot{\gamma}(s)$ lies in the tangent space $T_{\gamma(t)}M$, the vector field Z in $(*)$ is evaluated at $\gamma(t)$.

(c) We have the following easy

Lemma 1 *If $a \neq 0$ is a constant and $W = aZ$, then the flow W^s of W satisfies*

$$W^s = Z^{as}.$$

Proof: Let $q \in M$ be arbitrary. Then

$$\frac{d}{ds} Z^{as}(q) = aZ(Z^{as}(q)) = W(Z^{as}(q)),$$

so $s \mapsto Z^{as}(q)$ is a solution to $\dot{x} = W(x)$ starting at q. Therefore, $Z^{as}(q) = W^s(q)$, by uniqueness. QED

Let $t \neq 0$ and $p \in M$ be fixed and consider the vector field

$$W = tZ.$$

By (b), γ is a solution to the ODE $\dot{x} = W(x)$ satisfying $\gamma(0) = p$, so $\gamma(s) = W^s(p)$, where W^s denotes the time-s map of the flow of W. By the Lemma, $\gamma(s) = Z^{ts}(p)$. Taking $s = t$ yields the desired conclusion.