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All manifolds are assumed to be of class $C^\infty$. 
1. (25 points) Let $f : M \to M$ be a diffeomorphism. If $f(p) = p$, $p$ is called a fixed point of $f$. A fixed point $p$ is called a Lefshetz fixed point if $1$ is not an eigenvalue of $T_pf : T_pM \to T_pM$. If all fixed points of $f$ are Lefshetz, $f$ is called a Lefshetz map.

(a) Show that a Lefshetz fixed point is isolated, i.e., it has a neighborhood $U$ such that $f$ has no other fixed points in $U$.

(b) If $M$ is compact and $f$ is a Lefshetz map, show that $f$ has finitely many fixed points.

Proof: (a) If $p$ is a Lefshetz fixed point of $f$, then for any coordinate neighborhood $(U, \varphi)$ of $p$, $\varphi(p)$ is a Lefshetz fixed point of $\tilde{f} = \varphi^{-1} \circ f \circ \varphi$. This is because

$$\tilde{f}_* = (\varphi_*)^{-1}f_*\varphi_*,$$

so $\tilde{f}_*$ and $f_*$ have the same eigenvalues. Therefore, it is enough to prove the statement of (a) for $M = \mathbb{R}^n$. So let $p$ is a Lefshetz fixed point of a diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ and consider the equation $f(x) = x$ in a neighborhood of $p$. We can rewrite it as

$$g(x) = 0,$$

where $g(x) = f(x) - x$. Since

$$Dg(p) = Df(p) - I,$$

and $1$ is not an eigenvalue of $Df(p)$, $Dg(p)$ is non-singular. By the Inverse Function Theorem, $g$ is a local diffeomorphism near $p$. That is, there exists a neighborhood $U$ of $p$ and a neighborhood $V$ of $0 = g(p)$ such that $g : U \to V$ is a diffeomorphism. In particular, the equation $g(x) = 0$ has a unique solution $p$ in $U$. This means that the only fixed point of $f$ in $U$ is $p$.

(b) Suppose that $(p_k)$ is a sequence of fixed points of $f$. By compactness of $M$, it has a convergent subsequence, $p_{k_i} \to p$, as $i \to \infty$. Since $f(p) = f(\lim p_{k_i}) = \lim f(p_{k_i}) = \lim p_{k_i} = p$, $p$ is a fixed point of $f$. But this is impossible, since all fixed points of $f$ are Lefshetz, hence isolated by (a).
2. **(25 points)** Recall that $G(k, n)$ denotes the Grassmann manifold of $k$-planes in $\mathbb{R}^n$.

(a) Show that $O(n)$ acts transitively on $G(k, n)$.

(b) Find the isotropy subgroup of the $k$-plane

\[ E = \text{span}\{e_1, \ldots, e_k\}, \]

where $e_i$ denotes the $i^{th}$ vector in the standard orthonormal basis for $\mathbb{R}^n$.

**Solution:** (a) We will show that the orbit of the $k$-plane $E$ defined in (b) is all of $G(k, n)$.

Let $F \in G(k, n)$ be arbitrary. Pick an orthonormal basis $\{f_1, \ldots, f_k\}$ of $F$ and using Gramm-Schmidt, extend it to an orthonormal basis $\{f_1, \ldots, f_n\}$ of $\mathbb{R}^n$. Let $A$ be the matrix whose columns are $f_1, \ldots, f_n$. By construction, $A \in O(n)$ and $Ae_i = f_i$, i.e., $AE = F$. This proves that the orbit of $E$ is $G(k, n)$, so the action is transitive.

(b) Suppose $A$ is in the isotropy group $G$ of $E$, i.e., $AE = E$. Since $A$ is an orthogonal matrix, $AE^\perp = E^\perp$, so $A$ is a block matrix of the form

\[
\begin{bmatrix}
P & 0 \\
0 & Q
\end{bmatrix},
\]

where $P \in O(E) \cong O(k)$ and $Q \in O(E^\perp) \cong O(n - k)$. Therefore, $G$ is isomorphic to $O(k) \times O(n - k)$. 

3. (25 points) Let $X$ be a $C^\infty$ vector field on a manifold $M$. Assume that $X$ is non-vanishing, i.e., $X(p) \neq 0$, for all $p \in M$ and let $f : M \to \mathbb{R}$ be an arbitrary $C^\infty$ function. Show that the equation

$$Xu = f$$

is locally solvable. That is, for every $p \in M$ there exists a neighborhood $U$ of $p$ and $u \in C^\infty(U)$ such that $Xu = f$.

**Proof:** By the Flow Box Theorem, $X$ locally looks like the vector field $\partial/\partial x_1$ and the equation $\partial u/\partial x_1 = f$ is clearly solvable. More precisely, for every $p \in M$ there exists a coordinate neighborhood $(U, \varphi)$ such that $\varphi(p) = 0$, $\varphi(U)$ is a cube centered at $0$, and

$$\varphi_*(X) = \frac{\partial}{\partial x_1}.$$  

Let $f \in C^\infty(M)$ be arbitrary. In this coordinate neighborhood, the PDE $Xu = f$ translates into

$$\frac{\partial u}{\partial x_1} = \tilde{f},$$  

where $\tilde{f} = f \circ \varphi^{-1}$. For $x = (x_1, \ldots, x_n) \in \varphi(U)$ define

$$\tilde{u}(x) = \int_0^{x_1} \tilde{f}(t, x_2, \ldots, x_n) \, dt.$$  

Then $\tilde{u}$ is a solution to (1) on $\varphi(U)$. Let $u = \tilde{u} \circ \varphi$. Clearly, $u \in C^\infty(U)$. Recalling that $\varphi_*(X)\tilde{u} = X(\tilde{u} \circ \varphi) = (Xu) \circ \varphi^{-1}$, we obtain

$$(Xu) \circ \varphi^{-1} = \varphi_* (X) \tilde{u}$$

$$= \frac{\partial \tilde{u}}{\partial x_1}$$

$$= \tilde{f}$$

$$= f \circ \varphi^{-1}.$$  

Since $\varphi : U \to \varphi(U)$ is a diffeomorphism, it follows that $Xu = f$ on $U$. 


4. (25 points) Prove the following statements.

(a) The map $\mathcal{P} : G(k, n) \to G(n - k, n)$ defined by $\mathcal{P}(E) = E^\perp$

is a diffeomorphism.

(b) The Grassmann manifold $G(2, 3)$ and the projective 2-space $P^2(\mathbb{R})$ are diffeomorphic.

Hint: Let $E$ be a $k$-plane in $\mathbb{R}^n$. Denote its orthogonal complement by $E^\perp$ and identify $\mathbb{R}^n$ with $E \times E^\perp$. There exists a neighborhood $\mathcal{U}$ of $E$ in $G(k, n)$ such that every $F \in \mathcal{U}$ is the graph of a unique linear map $L : E \to E^\perp$. Define $\varphi : \mathcal{U} \to \mathcal{L}(E, E^\perp)$ by $\varphi(F) = L$, where $\mathcal{L}(E, E^\perp)$ denotes the space of linear maps $E \to E^\perp$. Then $\{(\mathcal{U}, \varphi)\}$ is a smooth structure on $G(k, n)$, equivalent to the one defined in Boothby. Feel free to use it.

Proof: (a) Since $(E^\perp)^\perp = E$, $\mathcal{P}$ is clearly 1–1 and onto, so it suffices to show that it is smooth.

Let $E \in G(k, n)$ be fixed. Identify $\mathbb{R}^n$ with $E \times E^\perp$, that is, identify $E$ with $\mathbb{R}^k \times \{0_{n-k}\}$ and $E^\perp$ with $\{0_k\} \times \mathbb{R}^{n-k}$. Every $w \in \mathbb{R}^n$ can be written in a unique way as the sum $u + v$, where $u \in E$ and $v \in E^\perp$.

Let $(\mathcal{U}, \varphi)$ be a coordinate neighborhood of $E$ in $G(k, n)$ as defined in the hint with $\varphi : \mathcal{U} \to \mathcal{L}(E, E^\perp)$. Let $\mathcal{V} = \mathcal{P}(\mathcal{U})$ be the corresponding coordinate neighborhood of $E^\perp$ in $G(n - k, n)$ and $\psi : \mathcal{V} \to \mathcal{L}(E^\perp, E)$ the corresponding coordinate map.

Let $F \in \mathcal{U}$ be arbitrary and set $L = \varphi(F)$. This means that $F$ is the graph of $L$, i.e.,

$$F = \{u + Lu : u \in E\}.$$

How do we express $F^\perp$ as the graph of some $K \in \mathcal{L}(E^\perp, E)$? We therefore want

$$F^\perp = \{Kv + v : v \in E^\perp\}.$$

We are therefore looking for a linear map $K : E^\perp \to E$ such that

$$(u + Lu) \cdot (Kv + v) = 0,$$

for all $u \in E$ and $v \in E^\perp$. Using $u \cdot vKv = Lu \cdot Kv = 0$, this simplifies to

$$u \cdot Kv = -Lu \cdot v.$$

Since $Lu \cdot v = u \cdot L^*v$, where $L^*$ is the adjoint (i.e., transpose) of $L$, it follows that $K = -L^*$. If we choose a basis of $E$ and $E^\perp$, e.g., the standard basis, the matrix of $K$ is just the negative transpose of the matrix of $L$. Since the map

$$\mathcal{P}(L) = -L^*$$

is linear, it is smooth. Therefore, $\mathcal{P}$ is smooth, hence a diffeomorphism.

(b) By definition $G(1, 3) = P^2(\mathbb{R})$ and by (a), $G(2, 3) \approx G(1, 3)$. 

**Extra credit (25 points)** Let $X$ and $Y$ be $C^\infty$ vector fields on a compact manifold $M$ and assume

$$[X, Y] = Z, \quad [X, Z] = 0, \quad [Y, Z] = 0.$$  

Denote the (time-$t$ maps of the) flows of $X, Y, Z$ by $X^t, Y^t, Z^t$, respectively.

(a) Show that

$$(X^t)_*(Y) = Y - tZ,$$

for all $t \in \mathbb{R}$.

(b) Let $t \in \mathbb{R}$ and $p \in M$ be fixed and define a curve $\gamma$ by

$$\gamma(s) = (Y^{-s} \circ X^{-t} \circ Y^s \circ X^t)(p).$$

Show that

$$\dot{\gamma}(s) = tZ(\gamma(s)),$$

for all $s \in \mathbb{R}$.

(c) Show that for all $t \in \mathbb{R}$ and $p \in M$.

$$(Y^{-t} \circ X^{-t} \circ Y^t \circ X^t)(p) = Z^{t^2}(p).$$

**Proof:** Recall that the assumptions $[X, Z] = 0$ and $[Y, Z] = 0$ imply the following

**Fact:** $X^t_*(Z) = Z$ and $Y^t_*(Z) = Z$.

(a) Let $p \in M$ be fixed and denote by $V(t)$ the value of $(X^t)_*(Y)$ at $p$. That is,

$$V(t) = T_{X^{-t}p}X^t(Y(X^{-t}p)).$$

Then $t \mapsto V(t)$ is a $C^\infty$ curve in $T_pM$ satisfying $V(0) = Y(p)$. Recall that

$$\left. \frac{d}{dt} \right|_{0} X^{-t}_*(Y) = L_X Y.$$

If we differentiate not at zero but at some $t$, then

$$\frac{d}{dt} X^{-t}_*(Y) = \left. \frac{d}{ds} \right|_{s=0} X^{-t+s}_*(Y)$$

$$= \left. \frac{d}{ds} \right|_{s=0} X^{-t} X^{-s}_*(Y)$$

$$= X^{-t}( \left. \frac{d}{ds} \right|_{s=0} X^{-s}_*(Y) )$$

$$= X^{-t}(L_X Y),$$
Therefore,  
\[ \dot{V}(t) = -X^{-t}(L_X Y) = -X_t(Z) = -Z, \]
evaluated at \( p \), of course. By the Fundamental Theorem of Calculus,  
\[ V(t) = V(0) + \int_0^t (-Z(p)) \, ds = Y(p) - tZ(p). \]

(b) By the Chain Rule, we have:  
\[ \dot{\gamma}(s) = -Y + Y^{-s}X^{-t}(Y) \]
\[ \overset{(a)}{=} -Y + Y^{-s}(Y + tZ) \]
\[ = -Y + Y^{-s}(Y) + tY^{-s}(Z) \]
\[ = -Y + Y + tZ \]
\[ = tZ. \]  

Here we used the fact that the vector field is invariant under its flow and \( Y^{-s}(Z) = Z \). Since \( \dot{\gamma}(s) \) lies in the tangent space \( T_{\gamma(t)}M \), the vector field \( Z \) in \( (*) \) is evaluated at \( \gamma(t) \).

(c) We have the following easy

**Lemma 1** If \( a \neq 0 \) is a constant and \( W = aZ \), then the flow \( W^s \) of \( W \) satisfies  

\[ W^s = Z^{as}. \]

**Proof:** Let \( q \in M \) be arbitrary. Then  
\[ \frac{d}{ds} Z^{as}(q) = aZ(Z^{as}(q)) = W(Z^{as}(q)), \]
so \( s \mapsto Z^{as}(q) \) is a solution to \( \dot{x} = W(x) \) starting at \( q \). Therefore, \( Z^{as}(q) = W^s(q) \), by uniqueness. QED

Let \( t \neq 0 \) and \( p \in M \) be fixed and consider the vector field  
\[ W = tZ. \]

By (b), \( \gamma \) is a solution to the ODE \( \dot{x} = W(x) \) satisfying \( \gamma(0) = p \), so \( \gamma(s) = W^s(p) \), where \( W^s \) denotes the time-\( s \) map of the flow of \( W \). By the Lemma, \( \gamma(s) = Z^{ts}(p) \). Taking \( s = t \) yields the desired conclusion.