Final Exam Solutions
Assigned on December 10, 2008

Due on December 17 by 12 noon

Name: Granwyth Hulatberi

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Explain your work
1. **(20 points)** Let \( \gamma(s) = (x(s), y(s), z(s)) \) be a unit speed curve such that \( \gamma(0) = (0, 0, 0) \) and its Frenet-Serret frame at \( s = 0 \) is

\[
    t = (1, 0, 0), \quad n = (0, 1, 0), \quad b = (0, 0, 1).
\]  

If \( \kappa \) and \( \tau \) are the curvature and torsion of \( \gamma \), show that

\[
    x(s) = s - \frac{\kappa^2}{6}s^3 + f(s), \\
    y(s) = \frac{\kappa}{2}s^2 + \frac{\kappa'}{6}s^3 + g(s), \\
    z(s) = \frac{\kappa\tau}{6}s^3 + h(s),
\]

for some functions \( f, g, h \) such that \( f(s)/s^3, g(s)/s^3, h(s)/s^3 \) all converge to zero, as \( s \to 0 \).

*(Hint: Use Taylor expansion of order three.)*

**Solution:** The order three Taylor expansion of \( \gamma \) at 0 is

\[
    \gamma(s) = \gamma(0) + \gamma'(0)s + \frac{\gamma''(0)}{2!}s^2 + \frac{\gamma'''(0)}{3!}s^3 + R(s),
\]

where \( R(s)/s^3 \to 0 \), as \( s \to 0 \). We need to compute \( \gamma'''(0) \). Since \( \gamma'''(s) = \kappa(s)n(s) \), we obtain

\[
    \gamma'''(0) = \kappa'(0)n(0) + \kappa(0)n'(0).
\]

Since \( n(0) = (0, 1, 0) \) and \( n' = -\kappa t + \tau b \) (by Frenet-Serret equations), we obtain

\[
    \gamma'''(0) = \kappa'(0, 1, 0) + \kappa\{-\kappa(1, 0, 0) + \tau(0, 0, 1)\}
    = (-\kappa^2, \kappa', \kappa\tau).
\]

Using \( \gamma(0) = (0, 0, 0) \), \( \gamma'(0) = t(0) \), and \( (1) \), we obtain

\[
    \gamma(s) = \left( s - \frac{\kappa^2}{6}s^3, \frac{\kappa}{2}s^2 - \frac{\kappa'}{6}s^3, \frac{\kappa\tau}{6}s^3 \right) + R(s),
\]

which implies the claim, with \( R(s) = (f(s), g(s), h(s)) \).
2. (a) **(20 points)** Find five non-isometric surfaces such that the unit circle $C$ in the $xy$-plane is a geodesic of each.

(b) **(Extra credit, 20 points)** Show that for every real number $k$ there exists a surface $S$ such that $C$ is a geodesic of $S$ and the Gaussian curvature of $S$ along $C$ equals $k$.

**Solution:** (b) Let $k \in \mathbb{R}$ be arbitrary and let us find a surface of revolution $S$ satisfying the requirements. Assume $S$ is obtained by rotating a unit speed curve $\gamma(t) = (f(t), 0, g(t))$ in the $xz$-plane about the $z$-axis, with $f > 0$ and $\gamma(0) = (1, 0, 0)$, i.e., $f(0) = 1$ and $g(0) = 0$. Then $C$ is clearly a parallel of $S$ and $C$ is a geodesic iff $f'(0) = 0$. The Gaussian curvature of $S$ along $C$ was shown to be equal $-f''/f$. Therefore, we are looking for a solution to the following second order initial value problem:

$$f'' + kf = 0, \quad f(0) = 1, \quad f'(0) = 0.$$

By the theory of ordinary differential equations, it is known that a solution exists on some neighborhood of zero and is unique. We obtain $g$ by solving the equation $(f')^2 + (g')^2 = 1$ with the constraint $g(0) = 0$. For each $k$ we obtain a surface $S_k$ such that $C$ is a geodesic of $S_k$. If $k \neq \ell$, then $S_k$ is not isometric to $S_\ell$ by Gauss’s Theorema Egregium.

(a) follows from (b).
3. (20 points) Let $a \neq 0$. Compute the Gaussian and mean curvatures at the origin $(0, 0, 0)$ of the hyperboloid

$$z = axy.$$ 

Solution: The hyperboloid (also known as a hyperbolic paraboloid) is covered by a single surface patch $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$,

$$\sigma(u, v) = (u, v, auv).$$ 

Differentiating, we obtain:

$$\sigma_u = (1, 0, av), \quad \sigma_v = (0, 1, au),$$

and

$$\sigma_{uu} = \sigma_{vv} = 0, \quad \sigma_{uv} = \sigma_{vu} = (0, 0, a).$$

Therefore,

$$\sigma_u \times \sigma_v = \begin{vmatrix} i & j & k \\ 1 & 0 & av \\ 0 & 1 & au \end{vmatrix} = (-av, -au, 1),$$

so the standard unit normal is

$$N = \frac{(-av, -au, 1)}{\sqrt{1 + a^2u^2 + a^2v^2}}.$$ 

It follows that the coefficients of the first and second fundamental forms are

$$E = 1 + a^2v^2, \quad F = a^2uv, \quad G = 1 + a^2u^2$$

and

$$L = 0, \quad M = \frac{a}{\sqrt{1 + a^2u^2 + a^2v^2}}, \quad N = 0.$$ 

Observe that at the origin, $F = 0$ and $M = a$. Therefore,

$$K = \frac{LN - M^2}{EG - F^2} = -a^2$$

and

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)} = 0.$$
4. **(20 points)** Show that a diffeomorphism between surfaces that is both conformal and area preserving is an isometry.

**Proof:** Denote such a diffeomorphism by \( f : S_1 \rightarrow S_2 \). Let \( \sigma \) be an arbitrary parametrization for \( S_1 \). Then \( \tilde{\sigma} = f \circ \sigma \) is a parametrization for \( S_2 \). It suffices to show that \( \sigma \) and \( \tilde{\sigma} \) have the same first fundamental form.

Denote their first fundamental forms by \( Edu^2 + 2Fdudv + Gdv^2 \) and \( \tilde{E}du^2 + 2\tilde{F}dudv + \tilde{G}dv^2 \), respectively. Since \( f \) is conformal,

\[
\tilde{E} = \lambda E, \quad \tilde{F} = \lambda F, \quad \tilde{G} = \lambda G,
\]

for some smooth function \( \lambda > 0 \). Since \( f \) is area preserving,

\[
\tilde{E}\tilde{G} - \tilde{F}^2 = EG - F^2.
\]

From \( \tilde{E}\tilde{G} - \tilde{F}^2 = \lambda^2(EG - F)^2 \), it follows that \( \lambda^2 = 1 \), so by positivity \( \lambda = 1 \). Therefore, the first fundamental forms of \( \sigma \) and \( \tilde{\sigma} \) are the same. This proves that \( f \) is an isometry.
5. (20 points) Recall that a regular curve \( C \) on a surface \( S \) is called a line of curvature if the tangent vector to \( C \) is a principal vector for \( S \) at all points of \( C \).

Suppose that two surfaces \( S_1, S_2 \) intersect along a smooth curve \( C \) that is a line of curvature of \( S_1 \). Show that \( C \) is a line of curvature of \( S_2 \) if and only if the angle between the tangent planes \( T_pS_1 \) and \( T_pS_2 \) is constant for all \( p \in C \). (Recall that the angle between two planes is the angle between their normals.)

**Proof:** Let \( \gamma \) be a parametrization of \( C \). Denote a unit normal of \( S_i \) at \( \gamma(s) \) by \( \hat{N}_i(s) \). Since \( C \) is a line of curvature of \( S_1 \), we have (see ex. 6.18 in Pressley)

\[
\dot{\hat{N}}_1 = -\lambda_1 \dot{\gamma},
\]

for some function \( \lambda_1 \).

(\( \Rightarrow \)) Assume \( C \) is a line of curvature of \( S_2 \). Then \( \dot{\hat{N}}_2 = -\lambda_2 \dot{\gamma} \), for some function \( \lambda_2 \). Observe that \( \hat{N}_i \cdot \dot{\gamma} = 0 \), for \( i = 1, 2 \). Therefore,

\[
\frac{d}{ds}(\hat{N}_1 \cdot \hat{N}_2) = \dot{\hat{N}}_1 \cdot \hat{N}_2 + \hat{N}_1 \cdot \dot{\hat{N}}_2 \\
= -\lambda_1 \dot{\gamma} \cdot \hat{N}_2 + \hat{N}_1 \cdot (\dot{\hat{N}}_2) \\
= 0.
\]

It follows that \( \hat{N}_1 \cdot \hat{N}_2 = \cos \angle(\hat{N}_1, \hat{N}_2) \) is constant, proving that the angle between the tangent planes is constant along \( \gamma \).

(\( \Leftarrow \)) Assume the angle between \( \hat{N}_1 \) and \( \hat{N}_2 \) is constant. Then

\[
0 = \frac{d}{ds}(\hat{N}_1 \cdot \hat{N}_2) \\
= \hat{N}_1 \cdot \hat{N}_2 + \hat{N}_1 \cdot \dot{\hat{N}}_2 \\
= -\lambda_1 \dot{\gamma} \cdot \hat{N}_2 + \hat{N}_1 \cdot \dot{\hat{N}}_2 \\
= \hat{N}_1 \cdot \dot{\hat{N}}_2.
\]

Therefore, \( \dot{\hat{N}}_2 \) is orthogonal to \( \hat{N}_1 \). Since \( \hat{N}_2 \) is a unit vector field, \( \dot{\hat{N}}_2 \) is also orthogonal to \( \hat{N}_2 \), hence it is proportional to \( \dot{\gamma} \), i.e., \( \dot{\hat{N}}_2 = -\lambda_2 \dot{\gamma} \), for some function \( \lambda_2 \). But this means that \( \gamma \) is a line of curvature of \( S_2 \).
6. (20 points) If $\sigma: U \to S$ is a parametrization of a surface $S$ and $f: S \to \mathbb{R}$ is a continuous function, the surface integral of $f$ over $S$ is defined by

$$\int \int_S f \, dA = \int \int_U f(u,v) \|\sigma_u \times \sigma_v\| \, dudv.$$ 

(a) Let $K$ be the Gaussian curvature of the torus $T^2$. Compute the surface integral

$$\int \int_{T^2} K \, dA.$$ 

(b) Interpret the answer in light of the Gauss-Bonnet theorem. (You are not allowed to use the Gauss-Bonnet theorem to compute the answer.)

Solution: (a) Using the standard parametrization $\sigma$ of $T^2$ as a surface of revolution (see class notes; recall that the curve that is being rotated about the $z$-axis is the circle of radius $b$ with center at $(a,0,0)$), we obtain

$$E = b^2, \quad F = 0, \quad G = (a + b \cos \theta)^2,$$

where $0 < b < a$ and $0 < \theta < 2\pi$. We computed the principal curvatures of $T^2$:

$$\kappa_1 = \frac{1}{b}, \quad \kappa_2 = \frac{\cos \theta}{a + b \cos \theta}.$$ 

The Gaussian curvature is

$$K = \kappa_1 \kappa_2 = \frac{\cos \theta}{b(a + b \cos \theta)},$$

and

$$\|\sigma_u \times \sigma_v\| = \sqrt{EG - F^2} = b(a + b \cos \theta).$$

Using Fubini’s theorem, we obtain:

$$\int \int_{T^2} K \, dA = \int_0^{2\pi} \int_0^{2\pi} \frac{\cos \theta}{b(a + b \cos \theta)} \sqrt{EG - F^2} \, d\theta d\phi
\begin{align*}
&= \int_0^{2\pi} \int_0^{2\pi} \cos \theta \, d\theta d\phi \\
&= \int_0^{2\pi} 0 \, d\phi \\
&= 0.
\end{align*}$$

(b) By the Gauss-Bonnet theorem,

$$\int \int_{T^2} K \, dA = 2\pi \chi(T^2) = 0,$$

since the Euler characteristic of $T^2$ is zero, for any choice of $a, b$. One can interpret this in the following way: the total amount of Gaussian curvature on any torus always has to equal zero.
If others would but reflect on mathematical truths as deeply and continuously as I have, they would make my discoveries.

*Carl Friedrich Gauß*

(I don’t believe this, but the quote at least indicates that Gauß was a humble man.)

**HAPPY HOLIDAYS AND ENJOY YOUR WINTER BREAK!**