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Explain your work
1. (20 points) Let a function $f$ be defined by

$$f(x, y) = \begin{cases} rac{x^4 + x^2 y^2 + y^4}{x^4 + x^2 y^2 + x y^2 + y^4} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0). \end{cases}$$

(a) Is $f$ continuous at the point $(x, y) = (2008\pi, e^{2008})$?

(b) Is $f$ continuous at the origin?

Justify your answers.

**Solution:** (a) $f$ is continuous at $(2008\pi, e^{2008})$ because $f$ is a rational function and the denominator of $f$ is non-zero at that point.

(b) No. Since

$$f(x, x) = \frac{3x^4}{5x^4} = \frac{3}{5} \rightarrow \frac{3}{5},$$

and $f(x, 0) = 1 \rightarrow 1$, as $x \rightarrow 0$, the limit of $f$ at $(0, 0)$ doesn’t exist and is therefore not equal $f(0, 0) = 1$. 
2. (20 points) Suppose that \( f \) is a differentiable function. If 
\[
z = y + f(x^2 - y^2),
\]
compute
\[
yz_x + xz_y.
\]

Solution: By the Chain Rule,
\[
z_x = f'(x^2 - y^2)2x \quad \text{and} \quad z_y = 1 + f'(x^2 - y^2)(-2y).
\]
Therefore,
\[
yz_x + xz_y = y[f'(x^2 - y^2)2x] + x[1 + f'(x^2 - y^2)(-2y)]
\]
\[
= x.
\]
3. **(20 points)** Show that the ellipsoid \( 3x^2 + 2y^2 + z^2 = 9 \) and the sphere

\[
x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0
\]

have a common tangent plane at the point \((1,1,2)\).

**Solution:** Let

\[
f(x, y, z) = 3x^2 + 2y^2 + z^2 = 9
\]

and

\[
g(x, y, z) = x^2 + y^2 + z^2 - 8x - 6y - 8z + 24.
\]

We have

\[
\nabla f(x, y, z) = \langle 6x, 4y, 2z \rangle
\]

and

\[
\nabla g(x, y, z) = \langle 2x - 8, 2y - 6, 2z - 8 \rangle,
\]

so \(\nabla f(1, 1, 2) = \langle 6, 4, 4 \rangle\) and \(\nabla g(1, 1, 2) = \langle -6, -4, -4 \rangle\). Since \(\nabla f(1, 1, 2) = -\nabla g(1, 1, 2)\), the tangent planes of the surfaces \(f = 0\) and \(g = 9\) have the same normal, hence coincide with each other.
4. (20 points) Find the maximum and minimum of

\[ f(x, y) = e^{-xy} \]

on the region \( D = \{(x, y) : x^2 + 4y^2 \leq 1\} \).

Solution: Interior of \( D \): Since

\[ f_x = -ye^{-xy} \quad \text{and} \quad f_y = -xe^{-xy}, \]

the only critical point of \( f \) in the interior of \( D \) is \((0, 0)\). Note that \( f(0, 0) = 1 \).

Boundary of \( D \): Let \( g(x, y) = x^2 + 4y^2 \). Since \( g_x = 2x \) and \( g_y = 8y \), the Lagrange equations are

\[ -ye^{-xy} = \lambda 2x, \quad (1) \]
\[ -xe^{-xy} = \lambda 8y, \quad (2) \]
\[ x^2 + 4y^2 = 1. \quad (3) \]

Multiplying (1) by \( x \) and (2) by \( y \) yields \( 2\lambda x^2 = 8\lambda y^2 \) or \( \lambda(x^2 - 4y^2) = 0 \). If \( \lambda = 0 \), then (1) and (2) imply \( x = y = 0 \), which contradicts (3). Therefore, \( \lambda \neq 0 \), so \( x^2 = 4y^2 \). Substituting into (3) yields

\[ 8y^2 = 1, \]

which means \( y = \pm \frac{1}{2\sqrt{2}} \). Since \( x^2 = 4y^2 \), this implies \( x^2 = 1/2 \), so \( x = \pm 1/\sqrt{2} \). This gives us four points:

\[ f\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right) = e^{1/4} \quad \text{and} \quad f\left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right) = f\left(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right) = e^{-1/4}. \]

Since \( e^{-1/4} < 1 < e^{1/4} \), it follows that \( e^{-1/4} \) is the minimum and \( e^{1/4} \) is the maximum of \( f \) on \( D \).
5. (20 points) Compute the double integral

\[ \int \int_D x \cos y \, dA, \]

where \( D \) is the region bounded by the curves \( y = 0, \ y = x^2, \) and \( x = \sqrt{\pi/2}. \)

**Solution:** Since \( D \) is a region of type I, by Fubini’s theorem we have:

\[
\int \int_D x \cos y \, dA = \int_0^{\sqrt{\pi/2}} \int_0^{x^2} x \cos y \, dy \, dx
\]

\[
= \int_0^{\sqrt{\pi/2}} (x \sin y) \bigg|_{y=0}^{y=x^2} \, dx
\]

\[
= \int_0^{\sqrt{\pi/2}} x \sin x^2 \, dx
\]

\[
= -\frac{1}{2} \cos x^2 \bigg|_0^{\sqrt{\pi/2}}
\]

\[
= -\frac{\cos \pi/2 - \cos 0}{2}
\]

\[
= \frac{1}{2}.
\]
6. (20 points) Let $H$ be the solid above the $xy$-plane bounded by the unit sphere $x^2 + y^2 + z^2 = 1$ and the $xy$-plane.

(a) Sketch $H$.

(b) Compute the triple integral 
\[
\iiint_H xyz \, dV.
\]

Solution: (a) The upper half of the unit ball.

(b) The projection of $H$ onto the $xy$-plane is the unit disk $D : x^2 + y^2 \leq 1$. Using Fubini’s theorem, we obtain:

\[
\iiint_H xyz \, dV = \int_D \int_0^{\sqrt{1-x^2-y^2}} xyz \, dz \, dA
\]

\[
= \int_D xy \frac{z^2}{2} \bigg|_{z=\sqrt{1-x^2-y^2}} dA
\]

\[
= \frac{1}{2} \int_D xy(1 - x^2 - y^2) \, dA.
\]

Here we use polar coordinates $(r, \theta)$ in which $D$ is described by

\[
0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.
\]

Thus the last integral equals:

\[
\int_0^{2\pi} \int_0^1 r^2 \cos \theta \sin \theta (1 - r^2) r \, dr \, d\theta = \int_0^{2\pi} \cos \theta \sin \theta \, d\theta \cdot \int_0^1 r^3(1 - r^2) \, dr
\]

\[
= 0,
\]

since

\[
\int_0^{2\pi} \cos \theta \sin \theta \, d\theta = \frac{1}{2} \int_0^{2\pi} \sin 2\theta \, d\theta
\]

\[
= 0.
\]