Name: Gottfried Wilhelm Newton

<table>
<thead>
<tr>
<th>Score</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
</tr>
<tr>
<td>2</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
</tr>
<tr>
<td>4</td>
<td>25</td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
</tr>
</tbody>
</table>

EXPLAIN YOUR WORK
1. **(25 points)** Find the limit if it exists or show that it does not exist:

$$
\lim_{(x,y) \to (0,0)} \frac{3x^2y^2}{3x^2 + 2y^2}.
$$

**Solution:** Since $3x^2 + 2y^2 \geq 3x^2$, for all $x$ and $y$, we obtain

$$
0 \leq \frac{3x^2y^2}{3x^2 + 2y^2} \leq \frac{3x^2y^2}{3x^2} = y^2.
$$

Both the left- and the right-hand side of the inequality converge to zero as $(x, y) \to (0, 0)$, so by the Squeeze Theorem the limit equals zero.
2. (25 points) Determine the set of points at which the given function is continuous:

\[ f(x, y) = \begin{cases} \frac{x^4}{x^4 + y^4} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0). \end{cases} \]

Solution: Away from the origin, \( f \) is a rational function hence continuous. Let us examine the continuity of \( f \) at the origin.

Let \( C_1 \) be the path defined by \( x = 0 \). Along \( C_1 \),

\[ f(x, y) = f(0, y) = 0 \to 0. \]

Let \( C_2 \) be the path defined by \( y = 0 \). Along \( C_2 \),

\[ f(x, y) = f(x, 0) = 1 \to 1. \]

Since the limits of \( f \) along \( C_1 \) and \( C_2 \) are different, it follows that \( f \) does not have a limit at \((0, 0)\) and is therefore discontinuous there.
3. **(25 points)** Compute the linearization of the function 

\[ f(x, y) = \arctan(x^2 + y) \]

at the point \((x, y) = (0, 0)\).

**Solution:** The linearization of \(f\) at \((0, 0)\) is the function 

\[ L(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y. \]

Clearly, \(f(0, 0) = \arctan 0 = 0\). By the Chain Rule,

\[
\begin{align*}
    f_x(x, y) &= \frac{2x}{1 + (x^2 + y)^2}, \\
    f_y(x, y) &= \frac{1}{1 + (x^2 + y)^2},
\end{align*}
\]

so \(f_x(0, 0) = 0, f_y(0, 0) = 1\). Therefore,

\[ L(x, y) = y. \]
4. (25 points) Suppose \( z = f(x, y) \) is a differentiable function, where, for \( s, t > 0 \),

\[
x = \frac{s}{t} \quad \text{and} \quad y = \frac{t}{s}.
\]

Show that

\[
sz_s + tz_t = 0.
\]

**Solution:** The Chain Rule gives

\[
z_s = z_x \frac{1}{t} + z_y \left( -\frac{t}{s^2} \right),
\]

\[
z_t = z_x \left( -\frac{s}{t^2} \right) + z_y \frac{1}{s}.
\]

Multiplying \( z_s \) by \( s \) and \( z_t \) by \( t \) and adding, we obtain

\[
sz_s + tz_t = z_x \frac{s}{t} - z_y \frac{t}{s} - z_x \frac{s}{t} + z_y \frac{t}{s} = 0.
\]