1. (Sec. 3.3, ex. 8) \(\Rightarrow\) Suppose \(f\) differentiable away from the origin and homogeneous of degree \(p\), i.e., \(f(tx) = t^p f(x)\), for all \(x \neq 0\) and \(t > 0\). Then differentiating at \(t = 1\) and using the (current version of) the chain rule, we obtain

\[
df(x)(x) = pf(x).
\]

(\(\Leftarrow\)) Now assume (1). Fix a nonzero \(x\) and define a function \(\psi(t) = t^{-p} f(tx)\). Differentiating at \(\psi(t)\) and using the chain rule, we obtain

\[
\psi'(t) = -pt^{-p-1} f(tx) + t^{-p} df(tx)(x).
\]

By (1), we obtain \(df(tx)(x) = t^{-1} df(tx)(tx) = t^{-1} pf(tx)\), which yields \(\psi'(t) = 0\), for all \(t > 0\). Therefore, \(\psi\) is constant, so for all \(t > 0\),

\[
t^{-p} f(tx) = \psi(t) = \psi(1) = f(x),
\]

proving that \(f\) is homogeneous of degree \(p\). \(\square\)

2. (Sec. 3.3, ex. 10) Let \(h = fg\) and define \(L = f(x_0) dg(x_0)\). We claim that \(dh(x_0) = L\). By definition of the derivative,

\[
\frac{g(x) - g(x_0) - dg(x_0)(x - x_0)}{|x - x_0|} \to 0, \quad \text{(2)}
\]
as \(x \to x_0\). We have:

\[
\frac{h(x) - h(x_0) - L(x - x_0)}{|x - x_0|} = \frac{f(x)g(x) - f(x_0)dg(x_0)(x - x_0)}{|x - x_0|}
\]
\[
= \frac{f(x)g(x) + f(x_0)g(x) - f(x_0)dg(x_0)(x - x_0)}{|x - x_0|}
\]
\[
= \frac{[f(x) - f(x_0)]g(x)}{|x - x_0|} + \frac{f(x_0)g(x) - g(x_0) - dg(x_0)(x - x_0)}{|x - x_0|}
\]
\[
A(x) + B(x).
\]

As \(x \to x_0\), \(B(x) \to 0\), by (2). Furthermore, since \(g(x_0) = 0\),

\[
A(x) = \frac{[f(x) - f(x_0)]g(x_0) - g(x_0)}{|x - x_0|}
\]
\[
= [f(x) - f(x_0)] \frac{g(x) - g(x_0) \pm dg(x_0)(x - x_0)}{|x - x_0|}
\]
\[
= [f(x) - f(x_0)] \frac{g(x) - g(x_0) - dg(x_0)(x - x_0)}{|x - x_0|} + [f(x) - f(x_0)] \frac{dg(x_0)(x - x_0)}{|x - x_0|}
\]
\[
\to 0 + 0 = 0,
\]
as \(x \to x_0\). The first term goes to zero by (2) and since \(f(x) \to f(x_0)\) (by continuity of \(f\)). The second term goes to zero because \(f(x) \to f(x_0)\) and

\[
|B(x)| \leq |f(x) - f(x_0)| \frac{|dg(x_0)(x - x_0)|}{|x - x_0|} \leq |f(x) - f(x_0)||dg(x_0)| \to 0,
\]
as \(x \to x_0\). Here we used the fact that for a linear function \(K\), \(|K(h)| \leq |K||h|\), for all vectors \(h\).
This proves that
\[
\frac{h(x) - h(x_0) - L(x - x_0)}{|x - x_0|} \to 0,
\]
which means that \(dh(x_0) = L\). □

3. (Sec. 3.4, ex. 1) We have:
\[
f(x, y, z) = xyz = [(x - 1) + 1][(y + 1) - 1]z
\]
\[
= -z + (x - 1)z - (y + 1)z + (x - 1)(y + 1)z.
\]
Notice that the right-hand side is a degree three Taylor-type polynomial at \(x_0 = (1, -1, 0)\). By uniqueness of the Taylor polynomial, it follows that this is the Taylor expansion of \(f\) at the point \(x_0\), for any \(q \geq 4\). □

4. (Sec. 3.4, ex. 2) (a) A calculation gives
\[
f_x(1, 0) = \frac{-\cos y}{x^2} \big|_{(1,0)} = -1, \quad f_y(1, 0) = -\frac{\sin y}{x} \big|_{(1,0)} = 0,
\]
and
\[
f_{xx}(1, 0) = 2 \frac{\cos y}{x^3} \big|_{(1,0)} = 2, \quad f_{xy}(1, 0) = f_{yx}(1, 0) = \frac{\sin y}{x^2} \big|_{(1,0)} = 0, \quad f_{yy}(1, 0) = -\frac{\cos y}{x} \big|_{(1,0)} = -1.
\]
Therefore, the Taylor expansion of \(f\) at \((1, 0)\) of order \(q = 2\) is
\[
f(x, y) = 1 - (x - 1) + R_2(x, y),
\]
where
\[
R_2(x, y) = \frac{1}{2!} \left\{ \frac{2 \cos \eta}{\xi^3} [x - 1]^2 + \frac{2 \sin \eta}{\xi^2} [x - 1]y + \frac{\cos \eta \xi^2}{\xi} \right\},
\]
for some point \((\xi, \eta)\) lying on the line segment joining \((x, y)\) and \((1, 0)\). Therefore, if \(x \geq a\) (for some \(0 < a < 1\)), all second order partials are bounded by \(C = 2/a^3\), so (by formula (3.15) with \(q = 2\))
\[
|R_2(x, y)| \leq \frac{C Q^q/2 |h|^q}{q!} \leq \frac{2}{a^3} [(x - 1)^2 - y^2].
\]
(b) It is not hard to see (by induction) that
\[
\frac{\partial^{i+j} f}{\partial x^i \partial y^j} = (\pm 1)^i \frac{\sin y \text{ or } \cos y}{x^{i+1}}.
\]
Thus if \(x \geq a\), for some \(0 < a < 1\), then
\[
\left| \frac{\partial^q f}{\partial x^a \partial y^b} \right| \leq \frac{i!}{a^{i+1}} \leq \frac{q!}{a^{q+1}},
\]
where \(q = i + j\). Therefore,
\[
|R_q(x, y)| = \left| \frac{1}{q!} \sum_{i+j=q} \frac{\partial^q f}{\partial x^i \partial y^j} (\xi, \eta) (x - 1)^i y^j \right|
\]
\[
\leq \frac{1}{q!} \sum_{i+j=q} \frac{q!}{a^{q+1}} |x - 1|^i |y|^j
\]
\[
= \frac{(|x - 1| + |y|)^q}{a^{q+1}}.
\]
Observe that \((|x - 1| + |y|)^q/a^{q+1} \to 0\), as \(q \to \infty\), if \(|x - 1| + |y| < a\). The inequality \(|x - 1| + |y| < a\) defines an open set \(U\) containing the point \((1, 0)\). Therefore, for all \((x, y) \in U\),
\[ R_q(x,y) \to 0, \text{ as } q \to \infty, \text{ i.e., } f \text{ is analytic on } U. \]

5. (Sec. 3.4, ex. 4) To show that \( f \) is of class \( C^1 \) on \( D \) it suffices to show that the partial derivatives of \( f \) exist and are continuous at \( x_0 \). Fix \( 1 \leq i \leq n \), and let \( t \neq 0 \). The Mean Value Theorem guarantees the existence of a point \( \xi_t \) on the line segment joining \( x_0 \) and \( x_0 + te_i \) such that

\[
\frac{f(x_0 + te_i) - f(x_0)}{t} = \partial_i f(\xi_t).
\]

Since \( \xi_t \to x_0 \), as \( t \to 0 \), it follows that the limit of the right-hand side exists as \( t \to 0 \), and equals \( \ell_i \). Therefore, the limit of the left-hand side also exists and by definition equals \( \partial_i f(x_0) \). This means that \( \partial_i f(x_0) = \ell_i = \lim_{x_0} \partial_i f \), proving that \( \partial_i f \) is continuous at \( x_0 \). Therefore, \( f \) is of class \( C^1 \) on the whole set \( D \).

The generalization of this result to higher-order partials is straightforward. \( \square \)

6 (Sec. 3.4, ex. 6) Let

\[
f(x) = \begin{cases} x^k \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}
\]

(a) Suppose \( k = 0 \). Since the limit of \( \sin(1/x) \) as \( x \to 0 \) doesn’t exist (Calculus I), it follows that \( f \) is not continuous at zero.

(b) Suppose \( k = 1 \). Since

\[
0 \leq |f(x)| \leq |x| \to 0 = f(0),
\]
as \( x \to 0 \), it follows that \( f \) is continuous at zero. \( f \) is continuous elsewhere as an elementary function. However,

\[
\frac{f(x) - f(0)}{x} = \sin \frac{1}{x}
\]
does not have a limit as \( x \to 0 \), so \( f \) is not differentiable at zero.

(c) Suppose \( k = 2 \). Then

\[
\frac{f(x) - f(0)}{x} = x \sin \frac{1}{x} \to 0,
\]
as \( x \to 0 \), so \( f \) is differentiable at zero. \( f \) is differentiable elsewhere as an elementary function. However, for \( x \neq 0 \),

\[
f'(x) = kx^{k-1} \sin \frac{1}{x} + x^k \cos \frac{1}{x} \left( -\frac{1}{x^2} \right) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.
\]
The first term converges to zero, as \( x \to 0 \), but the second term does not have a limit at zero. Therefore, \( f'(x) \not\rightarrow f'(0) \), as \( x \to 0 \), so \( f' \) is not continuous at zero. This means that \( f \) is not \( C^1 \).

(d) By induction, for any \( k \geq 3 \), \( f \) is \((k-1)\)-times differentiable everywhere, but not of class \( C^{k-1} \) as \( f^{(k-1)} \) fails to be continuous at zero. \( \square \)