1. (Sec. 4.1, ex. 1) The matrix of $L$ is
\[
[L] = \begin{bmatrix}
1 & 1 & 5 \\
-2 & 0 & 1
\end{bmatrix}.
\]
Since $\det \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix} = 2 \neq 0$, $L$ has rank 2. The kernel of $L$ can be computed by solving the system
\[
\begin{align*}
x_1 + x_2 + 5x_3 &= 0 \\
-2x_1 + x_3 &= 0.
\end{align*}
\]
All solutions are scalar multiples of $(1, -11, 2)$. Therefore,
\[
\text{Ker}(L) = \{(t, -11t, 2t) : t \in \mathbb{E}1\}.
\]

2. (Sec. 4.2, ex. 1) (a) Since $L(x, y) = (y, x)$, it follows that $L$ is the reflection with respect to the line $x = y$.

(b) We have:
\[
[SL] = [S][L] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\]
and
\[
[LS] = [L][S] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\]
In other words, $(SL)(x, y) = (y, -x)$ and $(LS)(x, y) = (-y, x)$. This means that $SL$ is the rotation by $-\pi/2$, whereas $LS$ is the rotation by $\pi/2$. Note that $LS = (SL)^{-1}$. \hfill \Box

3. (Sec. 4.2, ex. 2) A simple calculation. $L$ is a rotation. The axis of rotation is the eigendirection of the unique real eigenvalue of $L$. \hfill \Box

4. (Sec. 4.3, ex. 5) I will only present the solution to the part of the problem that refers to the function
\[
g(s, t) = (|s - t|, |s + t|).
\]
(a) Since the function $x \mapsto |x|, \mathbb{E}1 \to \mathbb{E}1$ is differentiable for $x \neq 0$, it follows that $g$ is differentiable on the set
\[
S = \{(s, t) : s \neq t, s \neq -t\}.
\]
(b) Figure 1 shows the regions where $s - t$ and $s + t$ have constant sign. If we write $(\text{sign}(s - t), \text{sign}(s + t)) = (\sigma_1, \sigma_2)$, then $(\sigma_1, \sigma_2) = (1, 1)$ on $A$, $(\sigma_1, \sigma_2) = (-1, 1)$ on $B$, $(\sigma_1, \sigma_2) = (-1, -1)$ on $C$ and $(\sigma_1, \sigma_2) = (1, -1)$ on $D$. It follows that on $A$,
\[
\partial_s g(s, t) = (1, 1), \quad \partial_t g(s, t) = (-1, 1).
\]
In general, on $S = A \cup B \cup C \cup D$, we have
\[
\partial_s g(s, t) = (\sigma_1, \sigma_2), \quad \partial_t g(s, t) = (-\sigma_1, \sigma_2).
(c, d) The Jacobian of $g$ on $S$ is

$$Jg(s,t) = \det Dg(s,t) = \begin{bmatrix} \sigma_1 & -\sigma_1 \\ \sigma_2 & \sigma_2 \end{bmatrix} = 2\sigma_1\sigma_2.$$ 

Since $Jg \neq 0$, it follows that the rank of $Dg$ is 2 at every point of $S$. □

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}

5. (Sec. 4.3, ex. 6) (a) Since

$$\frac{g(t) - g(t_0) - L(t - t_0)}{|x - x_0|} = 0 \rightarrow 0,$$

it follows that $Dg(t_0) = L$, for every point $t_0$.

(b) Suppose $Dg(t) = L$, for all $t \in \Delta$. Define $G(t) = g(t) - L(t)$. Then $DG(t) = 0$, for all $t \in \Delta$. Denote the components of $G$ by $G_1, \ldots, G_n$. Then $dG_i = 0$ on $\Delta$, for all $i$. Since $\Delta$ is connected, it follows that $G_i$ is constant on $\Delta$. Let $G_i(t) \equiv c_i$. It follows that $G(t) \equiv c = (c_1, \ldots, c_n)$, which means that

$$g(t) = L(t) + c.$$ 

Therefore, $g$ is affine. □