You are allowed to use the literature but not talk to each other.
1. **(25 points)** Suppose $f : \mathbb{E}^n \to \mathbb{E}^1$ is a continuous function.

   (a) If $f(x) \to \infty$, as $|x| \to \infty$, show that for any number $c$, the set $K = \{ x \in \mathbb{E}^n : f(x) \leq c \}$ is compact.

   (b) Let $C$ be a connected subset of $\mathbb{E}^n$ and assume that $f$ takes only integer values on $C$. Show that $f$ is constant.

**Proof:**

(a) Since $K = f^{-1}((-\infty, c])$ and $(-\infty, c]$ is closed in $\mathbb{E}^1$, it follows that $K$ is closed. Suppose that $K$ is unbounded. Then there exits a sequence $(x_k)$ in $K$ such that $|x_k| \to \infty$, as $k \to \infty$. By assumption, $f(x_k) \to \infty$, which contradicts the fact that $x_k \in K$, for all $K$. Therefore, $K$ is closed and bounded, hence compact.

(b) Since $C$ is connected, it follows that $f(C) \subset \mathbb{E}^1$ is connected as well. Thus $f(C)$ is an interval. On the other hand, $f(C) \subset \mathbb{Z}$. The only closed interval contained in $\mathbb{Z}$ is a single point set, which implies that $f$ is constant.
2. (25 points) Recall that for a twice differentiable real-valued function $f$ on $\mathbb{E}^n$, the Laplacian of $f$ is defined by

$$\Delta f = \sum_{i=1}^{n} \partial_{ii} f.$$ 

Suppose that $f : \mathbb{E}^n \to \mathbb{E}^1$ is $C^2$ and let $A = [a_{ij}]$ be an orthogonal matrix ($A^T A = I$). Define $g(x) = f(Ax)$. Show that

$$\Delta g(x) = \Delta f(Ax).$$

In other words, show that the Laplacian is invariant under orthogonal transformations of $\mathbb{E}^n$.

**Proof:** By the chain rule, $dg(x) = df(Ax)A$. This yields

$$\partial_j g(x) = \sum_{i=1}^{n} \partial_i f(Ax) a_{ij}.$$ 

Differentiating with respect to $x_j$, we obtain

$$\partial_{jj} g(x) = \sum_{i=1}^{n} \sum_{k=1}^{n} \partial_{ki} f(Ax) a_{kj} a_{ij}.$$ 

Therefore,

$$\Delta g(x) = \sum_{j} \partial_{jj} g(x)$$

$$= \sum_{i,j,k} \partial_{ki} f(Ax) a_{kj} a_{ij}$$

$$= \sum_{i,k} \partial_{ki} f(Ax) \sum_{j} a_{ij} a_{kj}$$

$$= \sum_{i,k} \partial_{ki} f(Ax) \sum_{j} (A)_{kj} (A^T)_{ji}$$

$$= \sum_{i,k} \partial_{ki} f(Ax) (AA^T)_{ki}$$

$$= \sum_{i,k} \partial_{ki} f(Ax) \delta_{ki}$$

$$= \sum_{i} \partial_{ii} f(Ax)$$

$$= \Delta f(Ax),$$

where $\delta_{ki}$ denotes the Kronecker delta.
3. (25 points) Using the method of Lagrange multipliers, find the constrained extrema of the function \( f(x, y) = x^2 + y^2 \) on the set

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,
\]

where \( 0 < a < b \).

**Solution:** Let \( \phi(x, y) = (x/a)^2 + (y/b)^2 - 1 \) and define \( F = f + \lambda \phi \). We are looking for solutions of the system of equations

\[
\nabla F(x, y) = 0, \quad \phi(x, y) = 0.
\]

We have \( F_x = 2x + 2\lambda x/a^2 \) and \( F_y = 2y + 2\lambda y/b^2 \), so \( F_x = 0 \) iff \( x = 0 \) or \( 1 + \lambda/a^2 = 0 \), and \( F_y = 0 \) iff \( y = 0 \) or \( 1 + \lambda/b^2 = 0 \). Note that \( x \) and \( y \) cannot be zero simultaneously. If \( x = 0 \), then \( y = \pm b \) and if \( y = 0 \), then \( x = \pm a \). Observe also that \( 1 + \lambda/a^2 \) and \( 1 + \lambda/b^2 \) cannot both be zero, since \( a \neq b \).

Therefore, we get four critical points of \( F \): \((\pm a, 0)\) and \((0, \pm b)\). Since \( f(\pm a, 0) = a^2 < b^2 = f(0, \pm b) \), \( f \) attains a constrained minimum at \((\pm a, 0)\) and a constrained maximum at \((0, \pm b)\).

**Remark.** Note that \( f \) is just the square of the distance from the origin, so the problem asks for the closest and farthest point on the ellipsoid \( \phi = 0 \) from the origin.
4. (25 points) Let \( f : \mathbb{E}^2 \to \mathbb{E}^1 \) be of class \( C^1 \), with \( f(2, -1) = -1 \). Set

\[
G(x, y, z) = f(x, y) + z^2, \quad H(x, y, z) = xz + 3y^3 + z^3.
\]

The equations

\[
G(x, y, z) = 0, \quad H(x, y, z) = 0.
\]

have the solution \((x, y, z) = (2, -1, 1)\).

(a) What conditions on the derivative of \( f \) ensure that there are \( C^1 \) functions \( x = g(y) \) and \( z = h(y) \) defined on an open set in \( \mathbb{E}^1 \) that satisfy both equations (1), such that \( g(-1) = 2 \) and \( h(-1) = 1 \)?

(b) Under the conditions of (a), and assuming \( df(2, -1) = \begin{bmatrix} 1 & -3 \end{bmatrix} \), find \( g'(-1) \) and \( h'(-1) \).

Solution: (a) Let \( F = (G, H) \) and define \( w = (x, z) \). Then

\[
\frac{\partial F}{\partial w} = \begin{bmatrix}
G_x & G_z \\
H_x & H_z
\end{bmatrix} = \begin{bmatrix}
f_x & 2z \\
z & x + 3z^2
\end{bmatrix},
\]

so if \( \det \frac{\partial F}{\partial w}(2, -1, 1) = (3z^2 f_x - 2z^2)(2, -1, 1) = 5f_x(2, -1) - 2 \neq 0 \), i.e., if \( f_x(2, -1) \neq 2/5 \), then by the Implicit Function Theorem, the equation \( F(x, y, z) = (0, 0) \) can be solved for \( x \) and \( z \) in terms of \( y \) in a neighborhood of \((2, -1, 1)\). The solution is given by a \( C^1 \) function \((x, z) = \phi(y) = (g(y), h(y))\) such that \( g(-1) = 2 \) and \( h(-1) = 1 \).

(b) Also by the Implicit Function Theorem, we have that

\[
\begin{bmatrix}
g'(-1) \\
h'(-1)
\end{bmatrix} = \phi'(-1) = -\left( \frac{\partial F}{\partial w}(2, -1, 1) \right)^{-1} \frac{\partial F}{\partial y}(2, -1, 1).
\]

Since

\[
\frac{\partial F}{\partial y}(2, -1, 1) = \begin{bmatrix} G_y(2, -1, 1) \\ H_y(2, -1, 1) \end{bmatrix} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}
\]

and

\[
\left( \frac{\partial F}{\partial w}(2, -1, 1) \right)^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 5 & -2 \\ -1 & 1 \end{bmatrix}
\]

we have

\[
\begin{bmatrix}
g'(-1) \\
h'(-1)
\end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 5 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 9 \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \end{bmatrix}.
\]
5. **(25 points)** Define a function $f$ by

$$f(x, y) = \frac{1}{1 - x^2 - y^2}.$$

(a) Find the domain $D$ of $f$ and show that $f$ is of class $C^\infty$ on $D$.

(b) Show that $f$ is analytic at $(0, 0)$, i.e., show that $f$ can be expanded into a Taylor series at $(0, 0)$ which converges to $f$ in some neighborhood $U$ of the origin. Find the largest such neighborhood $U$.

(c) Compute $\partial_1^4 \partial_2^3 f(0, 0) = \frac{\partial^7 f}{\partial x^4 \partial y^3}(0, 0)$.

**Solution:**

(a) The domain of $f$ is the plane minus the unit circle. $f$ is $C^\infty$ because it is rational.

(b) If $x^2 + y^2 < 1$, then

$$f(x, y) = \sum_{n=0}^{\infty} (x^2 + y^2)^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} x^{2k} y^{2(n-k)}.$$

This proves that $f$ is analytic at $(0, 0)$. The largest neighborhood on which this series converges is the open unit disk $U = \{(x, y) : x^2 + y^2 < 1\}$.

(c) Since all the powers of $x$ and $y$ in the above series are even, it follows that

$$\partial_1^4 \partial_2^3 f(0, 0) = 0.$$
6. (25 points) Let $Q = (0, \infty)^2$ be the first quadrant of $\mathbb{E}^2$ and define a map $\phi : Q \to Q$ by

$$\phi(u, v) = \left( \frac{u}{v}, uv \right).$$

(a) Show that $\phi$ is a diffeomorphism.

(b) Let $B$ be the portion of $Q$ lying between the hyperbolas $xy = 1$ and $xy = 2$ and the two straight lines $y = x$ and $y = 4x$. Sketch $B$ and find the set $A \subset Q$ such that $B = \phi(A)$.

(c) Evaluate the integral $\int_B x^2 y^3$.

**Solution:** (a) Both components of $\phi$ are $C^\infty$ on $Q$, so $\phi$ is $C^\infty$ on $Q$. Solving $\phi(u, v) = (x, y)$ for $u, v$, we obtain

$$\phi^{-1}(x, y) = \left( \sqrt{xy}, \sqrt{\frac{y}{x}} \right).$$

Since $\phi^{-1}$ is also $C^\infty$ on $Q$, it follows that $\phi$ is a $C^\infty$ diffeomorphism of $Q$.

(b) Applying $\phi^{-1}$ to the boundary of $B$, we obtain

$$A = [1, \sqrt{2}] \times [1, 2].$$

See Figure 1.

![Figure 1: $\phi$ is a diffeomorphism from $A$ to $B$.](image)

(c) We will use the change of variables theorem. Observe that

$$\det D\phi(u, v) = \det \begin{bmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{bmatrix} = 2\frac{u}{v} > 0$$
on $Q$. Thus

$$\int_B x^2 y^3 = \int_A \frac{u^2}{v^2} (uv)^3 \frac{2u}{v}$$

$$= \int_A 2u^6$$

$$= \int_1^2 \int_1^{\sqrt{2}} 2u^6 \, dudv$$

$$= 2 \int_1^{\sqrt{2}} u^6 \, du$$

$$= \frac{2}{7} (8\sqrt{2} - 1).$$