Name: Granwyth Hulatberi

| Score |  
|-------|---
| 1     | 25  
| 2     | 25  
| 3     | 25  
| 4     | 25  
| Total | 100 |

Explain your work
1. (25 points) Find the limit of $f$ at the origin if it exists, where:

(a) \[ f(x, y) = \frac{\sin(x^2 + y^2)}{\sqrt{x^2 + y^2}}. \]

(b) \[ f(x, y, z) = \frac{(xyz)^2}{(xyz)^2 + (x + y - z)^2}. \]

Solution: (a) Using the inequality $|\sin t| \leq |t|$, for all real numbers $t$, we obtain:

\[ 0 \leq |f(x, y)| \leq \frac{x^2 + y^2}{\sqrt{x^2 + y^2}} = \sqrt{x^2 + y^2}. \]

Since $\sqrt{x^2 + y^2} \to 0$, as $(x, y) \to (0, 0)$, it follows that $f(x, y) \to 0$ as well.

(b) Along the $x$-axis, $f$ equals to zero, hence converges to zero as $(x, y, z) \to (0, 0, 0)$. At any point $(x, y, z)$ on the plane $x + y - z = 0$ such that $xyz \neq 0$ (that is, away from the coordinate planes), $f$ equals 1, hence converges to 1, as $(x, y, z) \to (0, 0, 0)$. Therefore, the limit of $f$ at the origin does not exist.
2. (25 points) Recall that a set $A$ is dense in a set $B$ if every point of $B$ is an accumulation point of $A$.

Suppose that $A$ is dense in $\mathbb{E}^n$ and $f : \mathbb{E}^n \to \mathbb{E}^m$ is continuous. If $f(a) = 0$, for all $a \in A$, show that $f(x) = 0$, for all $x \in \mathbb{E}^n$.

Solution: Let $x \in \mathbb{E}^n$ be arbitrary. Since $A$ is dense in $\mathbb{E}^n$, $x$ is an accumulation point of $A$. This implies that there exists a sequence $(a_k)$ in $A$ such that $a_k \to x$, as $k \to \infty$. By continuity of $f$ (and a homework exercise),

$$\lim_{k \to \infty} f(a_k) = f(x).$$

But $f(a_k) = 0$, for all $k$, by assumption. Therefore, $f(x) = \lim 0 = 0$, as claimed.
3. (25 points) Suppose \( f : \mathbb{E}^n \to \mathbb{E}^1 \) is a continuous function, \( f(x) > 0 \), for all \( x \neq 0 \), and

\[
f(\alpha x) = \alpha f(x),
\]

for all \( \alpha > 0 \) and \( x \in \mathbb{E}^n \). Show that there exist numbers \( a, b > 0 \) such that for every \( x \in \mathbb{E}^n \),

\[
a |x| \leq f(x) \leq b |x|.
\]

Solution: Denote by \( S \) the unit sphere centered at the origin in \( \mathbb{E}^n \). Since \( S \) is closed and bounded, it is compact, by the Heine-Borel theorem. Therefore, \( f \), being continuous, has a minimum \( a \) and maximum \( b \) on \( S \); i.e., \( a \leq f(p) \leq b \), for all \( p \in S \).

If \( x = 0 \), (2) is clearly satisfied. So let \( x \neq 0 \) be an arbitrary point. Then \( p = x/|x| \) lies on \( S \), hence \( a \leq f(p) \leq b \). Using (1), we obtain:

\[
f(x) = f \left( |x| \frac{x}{|x|} \right) = |x| f(p),
\]

which implies

\[
a |x| \leq f(x) \leq b |x|.
\]
4. **(25 points)** Let \( f : \mathbb{E}^3 \rightarrow \mathbb{E}^1 \) be defined by

\[
f(x, y, z) = |x - y + 2z|.
\]

If \( f(x_0, y_0, z_0) = 0 \), find all directions \( v \) such that \( f \) has a derivative at \((x_0, y_0, z_0)\) in the direction of \( v \).

**Solution:** Denote by \( \Pi \) the plane defined by \( x - y + 2z = 0 \). We claim that the derivative of \( f \) at \( p_0 = (x_0, y_0, z_0) \) in the direction of \( v \) exists if and only if \( v \in \Pi \).

Suppose first that \( v \in \Pi \). Since \( \Pi \) is a linear subspace of \( \mathbb{E}^3 \), \( p_0 + tv \in \Pi \), for all real \( t \), so

\[
\lim_{t \to 0} \frac{f(p_0 + tv) - f(p_0)}{t} = 0 \rightarrow 0,
\]

as \( t \to 0 \). In other words, \( D_v f(p_0) = 0 \).

Suppose now that \( v \notin \Pi \). Write \( v = (a, b, c) \). Then \( a - b + 2c \) is either positive or negative. Assume the former: \( a - b + 2c > 0 \). Then

\[
\frac{f(p_0 + tv) - f(p_0)}{t} = \frac{|x_0 - y_0 + 2z_0 + t(a - b + 2c)| - 0}{t} = \frac{|t|(a - b + 2c)}{t} = \begin{cases} a - b + 2c & \text{if } t > 0 \\ -a + b - 2c & \text{if } t < 0. \end{cases}
\]

Therefore, the limit of the above difference quotient as \( t \to 0 \) does not exist, which means that \( f \) does not have a derivative at \( p_0 \) in the direction of \( v \).

The case \( a - b + 2c < 0 \) is handled analogously.