San José State University

Math 132, Fall 2009

Midterm 2

Assigned on November 5, 2009, due November 12, 2009.

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Explain your work
1. **(25 points)** Let \( \phi : \mathbb{E}^1 \to \mathbb{E}^1 \) be a \( C^q \) function \( (q \geq 2) \) and define \( f : \mathbb{E}^2 \to \mathbb{E}^1 \) by

\[
f(x, y) = \phi(ax + by),
\]

where \( a, b \) are real numbers. Show that the Taylor expansion of \( f \) of order \( q \) about the point \((0, 0)\) is

\[
f(x, y) = \sum_{n=0}^{q-1} \frac{\phi^{(n)}(0)}{n!} \sum_{i=0}^{n} \binom{n}{i} (ax)^i(by)^{n-i} + R_q(x, y).
\]

**Solution:**
2. **(25 points)** Let 

\[ f(x, y) = (y - x^2)(y - 3x^2). \]

(a) Show that \( f \) has a unique critical point (call it \( p \)).

(b) Show that the restriction of \( f \) to each line passing through \( p \) has a strict local minimum at \( p \).

(c) Show that \( f \) does not have an extremum at \( p \).

**Solution:**
3. (25 points) Let $H : \mathbb{E}^2 \to \mathbb{E}^1$ be a $C^2$ function and define $F : \mathbb{E}^2 \to \mathbb{E}^2$ by

$$F(x, y) = \left( \frac{\partial H}{\partial y}(x, y), -\frac{\partial H}{\partial x}(x, y) \right).$$

Assume that a curve $\gamma : \mathbb{E}^1 \to \mathbb{E}^2$ satisfies

$$\gamma'(t) = F(\gamma(t)), \quad (1)$$

for all $t \in \mathbb{E}^1$. Show that $t \mapsto H(\gamma(t))$ is constant.

Remark. The equation (1) is called a Hamiltonian system and $H$ is the corresponding Hamiltonian function or energy (of a mechanical system modeled by (1)). The problem is a mathematical statement of the physical law of conservation of energy.

Solution:
4. (25 points) Define $f : \mathbb{E}^2 \to \mathbb{E}^2$ by

$$f(x, y) = (e^x \cos y, e^x \sin y).$$

(a) Show that $f$ is differentiable at every point and compute its derivative.
(b) Show that the Jacobian of $f$ is not zero at any point on $\mathbb{E}^2$.
(c) Show that $f$ is not 1–1.
(d) Find the range $f(\mathbb{E}^2)$ of $f$.

Solution: