Name: Granwyth Hulatberi

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Explain your work
1. **(20 points)** Suppose \( \phi : \mathbb{R} \to \mathbb{R} \) is a differentiable function and for all \( x, y \neq 0 \), define

\[
f(x, y) = \phi \left( \frac{x}{y} \right) + \phi \left( \frac{y}{x} \right).
\]

Compute

\[
x f_x + y f_y.
\]

**Solution:** By the Chain Rule:

\[
f_x = \phi' \left( \frac{x}{y} \right) \frac{1}{y} - \phi \left( \frac{y}{x} \right) \frac{y}{x^2},
\]

\[
f_y = -\phi' \left( \frac{x}{y} \right) \frac{x}{y^2} + \phi' \left( \frac{y}{x} \right) \frac{1}{x},
\]

so

\[
x f_x + y f_y = x \phi' \left( \frac{x}{y} \right) \frac{1}{y} - x \phi \left( \frac{y}{x} \right) \frac{y}{x^2} \\
- y \phi' \left( \frac{x}{y} \right) \frac{x}{y^2} + y \phi' \left( \frac{y}{x} \right) \frac{1}{x}
\]

\[
= 0.
\]
2. (20 points) Find the absolute extrema of the function 
\[ f(x, y) = x^4 + 2x^2y^2 + y^4 + 1 \]
on the set \( D = \{(x, y) : x^2 + 4y^2 \leq 4\} \). Sketch \( D \).

Solution: Interior of \( D \): To find the critical points we solve
\[
\begin{align*}
    f_x &= 4x^3 + 4xy^2 = 4x(x^2 + y^2) = 0, \\
    f_y &= 4x^2y + 4y^3 = 4y(x^2 + y^2) = 0.
\end{align*}
\]
The only critical point of \( f \) is therefore \((0, 0)\).

The boundary of \( D \): Let \( g(x, y) = x^2 + 4y^2 \). Using Lagrange multipliers, we obtain the following equations:
\[
\begin{align*}
    4x^3 + 4xy^2 &= \lambda 2x \quad (1) \\
    4x^2y + 4y^3 &= \lambda 8y \quad (2) \\
    x^2 + 4y^2 &= 4. \quad (3)
\end{align*}
\]
From (1), we obtain \( x(2x^2 + 2y^2 - \lambda) = 0 \), so \( x = 0 \) or \( \lambda = 2x^2 + 2y^2 \). In the first case, using (3) we obtain \( y = \pm 1 \). In the second case and using (2), which is equivalent to \( y(x^2 + y^2 - 2\lambda) = 0 \), we obtain \( y = 0 \). This and (2) imply \( x = \pm 2 \). Therefore, the important points on the boundary are \((0, \pm 1)\) and \((\pm 2, 0)\).

Since
\[ f(0, 0) < f(0, \pm 1) < f(\pm 2, 0), \]
it follows that the absolute minimum of \( f \) on \( D \) is \( f(0, 0) = 1 \) while its absolute maximum on \( D \) is \( f(\pm 2, 0) = 17 \).
3. **(20 points)** Compute the double integral

\[
\iint_D e^{-x^2-y^2} \, dA,
\]

where \(D\) is bounded by the \(y\)-axis and the right semi-circle of \(x^2 + y^2 = 4\). Sketch \(D\).

**Solution:** In polar coordinates \(D\) can be described by

\[
0 \leq r \leq 2, \quad -\pi/2 \leq \theta \leq \pi/2.
\]

Therefore:

\[
\iint_D e^{-x^2-y^2} \, dA = \int_{-\pi/2}^{\pi/2} \int_0^2 e^{-r^2} r \, dr \, d\theta \\
= \pi \left( -\frac{1}{2} e^{-r^2} \right) \bigg|_0^2 \\
= \frac{\pi}{2}(1 - e^{-4}).
\]
4. (20 points) Compute the volume of the solid $E$ bounded by the surfaces

$$z = x^2 + y^2 \quad \text{and} \quad z = 36 - 3x^2 - 3y^2.$$ 

Sketch $E$.

**Solution:** The given surfaces are circular paraboloids. They intersect along the set $x^2 + y^2 = 36 - 3x^2 - 3y^2$, which is the circle

$$x^2 + y^2 = 36, \quad z = 9.$$ 

The projection to the $xy$-plane of the solid $E$ is therefore the disk $D : x^2 + y^2 \leq 9$, which in polar coordinates can be described by $0 \leq r \leq 3, 0 \leq \theta \leq 2\pi$. Therefore the volume of $E$ equals:

$$\iiint_E dV = \iint_D \int_{x^2+y^2}^{36-3x^2-3y^2} dz \, dA$$

$$= \iint_D (36 - 4x^2 - 4y^2) \, dA$$

$$= \int_0^{2\pi} \int_0^3 (36 - 4r^2)r \, dr \, d\theta$$

$$= 2\pi (18r^2 - r^4)|_0^3$$

$$= 162\pi.$$
5. (20 points) Let

\[ F(x, y) = \left( \frac{x}{1 + x^2 + y^2}, \frac{y}{1 + x^2 + y^2} \right) =: \langle P(x, y), Q(x, y) \rangle. \]

(a) Show that \( F \) is a conservative vector field.
(b) Find a function \( f \) such that \( F = \nabla f \).
(c) Compute \( \int_C P \, dx + Q \, dy \), where \( C \) is the curve \( \pi x^{10} + ey^{10} = \pi^6 \) oriented counterclockwise.

**Solution:** (a) Denote the components of \( F \) by \( P, Q \), respectively. Then

\[ P_y = -\frac{2xy}{(1 + x^2 + y^2)^2} = Q_x. \]

Since \( F \) is defined on all of \( \mathbb{R}^2 \), which is simply connected, it follows that \( F \) is conservative.

(b) We solve the following system of PDEs:

\[ \begin{align*}
    f_x &= \frac{x}{1 + x^2 + y^2} \quad (4) \\
    f_y &= \frac{y}{1 + x^2 + y^2}. \quad (5)
\end{align*} \]

Integrating (4) with respect to \( x \), we obtain

\[ f(x, y) = \frac{1}{2} \log(1 + x^2 + y^2) + \phi(y), \]

for some function \( \phi \). Differentiating with respect to \( y \) and equating with (5), we obtain

\[ \frac{y}{1 + x^2 + y^2} = \frac{y}{1 + x^2 + y^2} + \phi'(y). \]

Therefore, \( \phi' = 0 \), so \( \phi \) is constant. It follows that

\[ f(x, y) = \frac{1}{2} \log(1 + x^2 + y^2) + \text{constant}. \]

(c) Since \( C \) is a closed curve and \( F \) is conservative, it follows that

\[ \int_C P \, dx + Q \, dy = 0. \]
6. **(20 points)** Compute the work done by the vector field

\[ \mathbf{F}(x, y) = \langle x^4 - y^2, y^3 - xy \rangle \]

as it moves an object along the circle \( C : (x - 1)^2 + y^2 = 1 \) oriented counterclockwise.

**Solution:** We will use Green's theorem. Let \( D \) be the disk bounded by \( C \). Then:

\[
\text{Work} = \int_C P \, dx + Q \, dy \\
= \int_D (Q_x - P_y) \, dA \\
= \int_D (-y - (-2y)) \, dA \\
= \int_D y \, dA \\
= \int_0^2 \int_{-u(x)}^{u(x)} y \, dy \, dx \\
= \int_0^2 0 \, dx \\
= 0,
\]

where \( u(x) = \sqrt{1 - (x - 1)^2} \).