Name: Granwyth Hulatberi

<table>
<thead>
<tr>
<th>Score</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
</tr>
<tr>
<td>2</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
</tr>
<tr>
<td>4</td>
<td>25</td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
</tr>
</tbody>
</table>

Explain your work
1. (25 points) Let \( u = f(x, y) \), where \( f \) is a differentiable function and

\[
x = e^s \cos t, \quad y = e^s \sin t.
\]

Show that

\[
(u_x)^2 + (u_y)^2 = e^{-2s}[(u_s)^2 + (u_t)^2].
\]

**Solution:** Observe that

\[
x_s = x, \quad x_t = -y, \quad y_s = y, \quad y_t = x,
\]

and \( x^2 + y^2 = e^{2s} \). We now apply the Chain Rule and obtain:

\[
u_s = u_x x_s + u_y y_s = u_x x + u_y y, \quad \text{and} \quad u_t = u_x x_t + u_y y_t = u_x (-y) + u_y x.
\]

Therefore,

\[
u_s^2 + u_t^2 = (u_x x + u_y y)^2 + (u_x (-y) + u_y x)^2
\]

\[
= (u_x^2 x^2 + 2xyu_x u_y + u_y^2 y^2) + (u_x^2 y^2 - 2xyu_x u_y + u_y^2 x^2)
\]

\[
= u_x^2 (x^2 + y^2) + u_y^2 (x^2 + y^2)
\]

\[
= u_x^2 e^{2s} + u_y^2 e^{2s}.
\]

Dividing both sides by \( e^{2s} \) yields the desired identity.
2. **(25 points)** Find the extrema of the function

\[ f(x, y) = x^3 - 12xy + 8y^3. \]

**Solution:** Let us first find the critical points by solving the equation \( \nabla f = 0 \):

\[
\begin{align*}
    f_x &= 3x^2 - 12y = 0, \\
    f_y &= -12x + 24y^2 = 0.
\end{align*}
\]

The second equation implies \( x = 2y^2 \). Substituting into the first equation gives \( y^4 = y \), i.e., \( y(y^3 - 1) = 0 \), so \( y = 0 \) or \( y = 1 \). If \( y = 0 \), then the second equation implies \( x = 0 \). If \( y = 1 \), the second equation gives \( x = 2 \). Therefore, the critical points are

\[ p_0 = (0, 0) \quad \text{and} \quad p_1 = (2, 1). \]

The second order partials of \( f \) are

\[
\begin{align*}
    f_{xx} &= 6x, \\
    f_{yx} &= -12, \\
    f_{yy} &= 48y,
\end{align*}
\]

so

\[
D(x, y) = \begin{vmatrix}
    6x & -12 \\
    -12 & 48y
\end{vmatrix} = 288xy - 144.
\]

Since \( D(0, 0) = -144 < 0 \), the Second Derivative Test implies that \( p_0 \) is a **saddle**.

Since \( f_{xx}(2, 1) = 12 > 0 \) and \( D(2, 1) = 288 \cdot 2 - 144 = 432 > 0 \), the Second Derivative Test implies that \( p_1 \) is a **local minimum**.
3. (25 points) Find the absolute maximum and minimum of the function

\[ f(x, y, z) = x + y + z \]

on the set \( S \) defined by \( x^2 + y^2 + z^2 = 1 \).

Solution: We will apply the method of Lagrange Multipliers and solve the equation

\[ \nabla f(x, y, z) = \lambda \nabla g(x, y, z). \]

Since

\[ \nabla f(x, y, z) = (1, 1, 1), \quad \text{and} \quad \nabla g(x, y, z) = (2x, 2y, 2z) \]

we have

\[
\begin{align*}
1 &= \lambda 2x \\
1 &= \lambda 2y \\
1 &= \lambda 2z \\
1 &= x^2 + y^2 + z^2.
\end{align*}
\]

Note that \( \lambda \) must be different from zero. Solving equations (1)–(3) for \( x, y, \) and \( z \) we obtain

\[ x = y = z = \frac{1}{2\lambda}. \]

Substituting into (4), we obtain

\[ 3 \cdot \frac{1}{4\lambda^2} = 1, \quad \text{so} \quad \lambda = \pm \frac{\sqrt{3}}{2}. \]

Therefore, \( x = y = z = 1/\sqrt{3} = \sqrt{3}/3 \), so \( f \) attains its extrema on \( S \) at the points

\[ p_\pm = \pm \left( \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right). \]

Since \( f(p_+) = \sqrt{3} > -\sqrt{3} = f(p_-) \), \( f \) has an absolute maximum at \( p_+ \) and an absolute minimum at \( p_- \).
4. **(25 points)** Compute the double integral

\[ I = \int\int_R \sin^2 x \cdot \cos y \, dA, \]

where \( R = \{(x, y) : 0 \leq x \leq \pi/2, \; 0 \leq y \leq \pi\} \).

**Solution:** By Fubini’s Theorem, we have

\[
I = \int_0^{\pi/2} \left( \int_0^\pi \sin^2 x \cdot \cos y \, dy \right) \, dx \\
= \left( \int_0^{\pi/2} \sin^2 x \, dx \right) \left( \int_0^\pi \cos y \, dy \right) \\
= 0,
\]

since

\[
\int_0^\pi \cos y \, dy = \sin y \bigg|_0^\pi = 0.
\]