1. Let $D$ be an open disc in $\mathbb{C}$ and suppose $f \in H(D)$. A boundary point $a$ of $D$ is called a regular point of $f$ if there exists $r > 0$ and a function $g$ holomorphic on $D(a, r)$ such that such that $g = f$ on $D \cap D(a, r)$. Otherwise, $a$ is called singular.

Show that the set of regular points of $f$ is open.

2. If $f \in H(D)$ and every point of the boundary $\partial D$ is a singular point, then $\partial D$ is said to be the natural boundary of $f$.

Let

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}.$$  

(a) Show that the radius of convergence of the given power series is 1.

(b) Show that $f(z^2) = f(z) - z$, for all $z \in \mathbb{D}$.

(c) Let $k, n$ be positive integers and let $a = \exp(2\pi ik/2^n)$. Show that $f$ is unbounded on every radius in $\mathbb{D}$ which ends at $a$. That is, $f(ra)$ is unbounded, as $r \to 1$.

(d) Show that the unit circle is the natural boundary of $f$.

3. Suppose that $(f, D)$ and $(g, D)$ are function elements, $P$ is a polynomial in two variables:

$$P(z, w) = \sum_{k=0}^{n} \sum_{l=0}^{k} a_{l,k-l} z^k w^{k-l},$$

and $P(f(z), g(z)) = 0$, for all $z \in D$. Suppose $(f, D)$ and $(g, D)$ can be analytically continued along a path $\gamma$, to $(f, D_1)$ and $(g_1, D_1)$. Prove that $P(f_1(z), g_1(z)) = 0$, for all $z \in D_1$. 