THE DRIPPING HANDRAIL MODEL: TRANSIENT CHAOS IN ACCRETION SYSTEMS

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ABSTRACT

We define and study a simple dynamical model for accretion systems, the “dripping handrail” (DHR). The time evolution of this spatially extended system is a mixture of periodic and apparently random (but actually deterministic) behavior. The nature of this mixture depends on the values of its physical parameters—the accretion rate, diffusion coefficient, and density threshold. The aperiodic component is a special kind of deterministic chaos called transient chaos. The model can simultaneously exhibit both the quasi-periodic oscillations and very low frequency noise that characterize the power spectra of fluctuations of several classes of accretion systems in astronomy. For this reason, our model may be relevant to many such astrophysical systems, including binary stars with accretion onto a compact object—white dwarf, neutron star, or black hole—as well as active galactic nuclei. We describe the systematics of the DHR’s temporal behavior by exploring its parameter space using several diagnostics: power spectra, wavelet “scalegrams,” and Lyapunov exponents (which characterize the degree of chaos inherent in the time evolution of a system). In addition, we note that for large accretion rates, the DHR has periodic modes; the effective pulse shapes for these modes—evaluated by folding the time series at the known period—bear a resemblance to the similarly determined shapes for some X-ray pulsars. The pulsing observed in some of these models may be such periodic-mode accretion and not be due to pure rotation as in the standard pulsar model.

Subject headings: accretions, accretion disks — binaries: close — galaxies: active — pulsars: general — X-rays: stars

1. Flickering Processes

We are interested in astronomical systems that evolve in an unpredictable way, as evidenced by a disordered or flickering light curve. Of special interest is disk accretion, as in low-mass X-ray binaries and active galactic nuclei, where the flow of gas is probably turbulent or otherwise disordered.

These systems almost certainly obey deterministic laws; their behavior only seems random. It is reasonable to suppose that complicated systems ought to be effectively unpredictable, but before the recently renewed interest in nonlinear dynamics, few physical models of such behavior were generally known. Quantitative models could be obtained only by assuming ad hoc properties of a postulated source of random behavior. Results in nonlinear dynamics have provided specific mechanisms that may be responsible for the apparently random behavior of even simple astrophysical systems. Furthermore, the fluctuations in time of many such models exhibit scaling or self-similar behavior—often referred to as 1/f noise. New analysis tools provided by, e.g., wavelet analysis (Daubechies 1992) and singularity spectral methods (Halsey et al. 1986; Mandelbrot 1974) are well suited for the study of both time series data from, and theoretical models for, this kind of physical system.

Keep in mind that the word noise is used for two quite different concepts: (1) stochastic or chaotic variability inherent to the source, and (2) random observational errors. The former is the subject of this paper and should be carefully distinguished from the latter.

This paper analyzes a model that is possibly relevant to a variety of astrophysical accretion systems. In Scargle et al. (1993) the dripping handrail (DHR), a simple deterministic model with apparently random behavior, was used to try to elucidate the luminosity fluctuations of low mass X-ray binaries (LMXBs), such as Sco X-1. This model provided a unified description of quasi-periodic oscillations (QPOs) and very low frequency noise (VLFN), which are the dominant components of the power spectra of these and other objects. In such systems, gas from a binary companion is assumed to be overflowing its Roche lobe and accreting onto a neutron star.

We propose the dripping handrail to describe the hot inner edge (“rail”) of the accretion disk. Matter accumulates on and diffuses along the edge. It falls off the rail—in blobs—and onto the surface of the compact star when a fixed density threshold is exceeded. The implementation of the dripping handrail we employ models the accumulation on the edge as a density that increases linearly in time. This is a simple, commonly used, and phenomenologically sufficient model for threshold-driven systems such as actual dripping handrails in which water condensation on the rail from a saturated vapor (fog) is modeled as a linear increase in water density. In that case, the threshold is a direct measure of the critical ratio of gravitational force to surface tension per unit length for a drop to be released from the rail.

We assume that the luminosity fluctuations are connected with the drops or blobs of hot gas, the sizes of which are distributed over a wide range. The observed luminosity could arise from the hot gas on the rail or as the gas plunges into the deep potential well of the central compact object or

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both. Since the main topic here is the phenomenology of the dynamical evolution of the DHR, we ignore the specifics of the radiation mechanism, which are discussed elsewhere (Imamura et al. 1993; Steiman-Cameron et al. 1994).

Much of the usefulness of the DHR model stems from its chaotic evolution in certain parameter regimes. This obviates the need to postulate an external source of randomness. Most of the dynamical models for disordered phenomena in the astrophysical literature postulate an ad hoc random element. For example, some of the references mentioned below assume that the randomness arises from fluctuations in the accretion rate. In a number of cases, linear analysis indicates the existence of instabilities that are presumed to yield apparently random dynamical behavior. Such deductions can be incorrect and may simply push our lack of understanding into a different box. For example, instabilities found from a local analysis can disappear when a global analysis is carried out (Chen & Taam 1995).

The DHR exhibits apparently random behavior even when the accretion rate is strictly constant (in both time and space) in the same way in which a nonlinear oscillator driven at a constant frequency can exhibit unpredictable behavior. Further, descriptors of this behavior (e.g., the Fourier and wavelet power spectra) are in agreement with observational results on the quasi-periodic oscillations and very low, frequency noise of at least Sco X-1 (Scargle et al. 1993). The DHR provides a simple explanation of both the QPOs and VLFN with a single physical model, invoking basic physical processes (diffusion, accretion, and a density threshold). In addition, for various accretion rates, the DHR produces other phenomenology that mirrors observational data on accreting systems, including strictly periodic modes and the appearance of subharmonics (or alternating high and low peaks) in periodic modes.

Recent use of the DHR to model accretion in the AM Herculis systems (Imamura et al. 1993; Steiman-Cameron et al. 1993) suggests that the basic DHR phenomenology may be widely useful in modeling astrophysical accretion, perhaps even in active galactic nuclei (AGNs). In the case of AM Her objects, the accretion is assumed to proceed along the field lines of a white dwarf with a large magnetic field, the processes of spatial diffusion and accretion with thresholding may be the dominant mechanisms, despite the fact that the geometry of the gas flow is most likely very different from that in LMXBs. Observational evidence at many wavelengths suggest that QPOs and VLFN are in fact generic properties of many accreting systems.

Here is a brief review of some recent work on related dynamical models of accretion. A simple “sandpiper” model (Mineshige, Ouchi, & Nishimori 1994) has some similarities to the dripping handrail. These authors postulated random accretion; we have carried out some numerical experiments that indicate that this is not needed, as the evolution from typical initial data is intrinsically stochastic—just as in the dripping handrail. Related models have been applied to X-ray transient sources that may be black holes (Mineshige 1994). The nonlinear evolution of thermal-viscous instabilities in the inner regions of accretion disks surrounding black holes and neutron stars has been studied (Chen & Taam 1994). These authors find periodic behavior at frequencies compatible with the observed QPOs and suggest that “Such oscillations may exhibit a quasiperiodic behavior if the mass flow rate entering the inner region of the disk is not strictly constant.” As referred to above, Chen & Taam (1995) have shown that waves propagating across the disk structure tend to quench instabilities that are present in correct, but strictly local, stability analyses. They do find that for sufficiently low accretion rates the computed light curves appear random (see their Figs. 4 and 6); the corresponding power spectra (Figs. 5 and 7) have a QPO feature roughly in accord with observations but also have a rather narrower peak at higher frequency (about 100 Hz) that to this point has not been observed. It should be noted, however, that many of the observations of these sources do not have good enough time resolution to detect such a high frequency. Some observations made by Ginga, EXOSAT, and HEAO have time resolutions of a millisecond or two, but many of the data from these and other satellites have considerably less resolution. The next generation of X-ray satellites, such as NASA’s X-Ray Timing Explorer (XTE) and the Naval Research Lab’s X-Ray Telescope for the Argos satellite (USA), will have improved timing capabilities. Several workers have studied related accretion problems from the point of view of “jumps” (see, e.g., Orlandini & Morfill 1992). Micheli (1977) investigated a polar cap drip model that is generically similar to the DHR.

Our paper is organized as follows: Section 2 physically defines the dripping handrail and discusses its basic properties. In subsections, we describe three different computational implementations of the basic model: a map lattice, a partial differential equation, and a cellular automaton. Section 3 examines the behavior of the DHR as a function of the values of its parameters. Comparison of results with the three different computational realizations of the DHR shows that the behavior is largely the same. Section 4 summarizes the conclusions and suggests further work. The Appendix gives some mathematical comparisons and dimensional relationships among the three versions.

2. THE DRIPPING HANDRAIL (DHR)

In this section we introduce the basic dripping handrail (DHR) model, an example of mathematical constructs that have been used for more than a decade to study inherently noisy physical systems (Crutchfield 1983; Keeler & Farmer 1986). Despite its rather abstract origin, the DHR, with slight modification, can represent several dynamical effects in an accretion disk, as follows: Matter flows onto the inner edge of the disk, thought of as the rail. There it is somehow radially supported (e.g., by radiation pressure and centrifugal force) but may diffuse tangentially, along the rail. Further, there is a density threshold above which matter falls off the rail and onto the star. This threshold is assumed to be the result of a plasma or fluid instability of some sort, but the details are not important here.

There is a competition between smoothing and roughening effects: diffusion smooths the distribution of matter, while the falling of drops of various sizes off the rail increases density variations at neighboring locations. We will show that the relative strength of these effects, and the time over which they act, is what determines the asymptotic and transient behavior of the system. (These effects lead to a model very much like, but even simpler than, the coupled, damped, driven oscillators that comprise the original DHR [Crutchfield & Kaneko 1988], as described in the next subsection.)

We will demonstrate that for DHR models with large numbers of degrees of freedom, the observed behavior is generally associated with the transient behavior of the
model. It has been argued elsewhere that this might be fairly ubiquitous behavior for nonlinear systems with many degrees of freedom (Crutchfield 1988). In fact, standard diagnostic measures (e.g., Lyapunov exponents) for stationary systems indicate that the behavior is chaotic. However, the DHR is chaotic during long-lasting transients rather than asymptotically. We use the term transient chaos to describe this behavior. We argue that this transient behavior is more important in practice than the asymptotic behavior. The reason is that variations in the accretion rate—regarded as external perturbations—will occur long before the system can reach its asymptotic state.

In the following three subsections, we present three different mathematical realizations of the basic physics just described. These differ only as to whether the physical variables—time, space, and matter density—are treated as continuous or discrete.

2.1. The Dripping Handrail as a Map Lattice

We start with the intermediate case, in which space and time are treated as discrete and the material density is represented as a continuous variable. We emphasize this version largely as a matter of convenience—because it is computationally very tractable and has been studied in the dynamical systems literature. One can think of such a representation as a discrete approximation to the fully continuous case. In the nonlinear dynamics literature such representations, called map lattices (MLs), are studied in their own right (Kaneko 1985; Keeler & Farmer 1986; Crutchfield & Kaneko 1988, 1987).

A common approach to map lattices is to postulate (1) that the system is spatially extended and composed of many elements or components, (2) that each such element satisfies an equation of motion expressible as a nonlinear map, and (3) that these elements interact with their neighbors via some kind of coupling. These map lattices (also referred to as coupled lattice maps or lattice dynamical systems) were proposed as a way to understand a very simple dynamical behavior thought to be nearly universal in spatially extended systems, namely elementary relaxation oscillators interacting with each other locally. For the most part, assumption (1) is not taken to reflect actual granularity but is meant in the same spirit in which a turbulent flow is represented in terms of finite sums of Fourier components. A recent example of the use of map lattice techniques is in the modeling of Rayleigh-Bénard convection (Yanagita & Kaneko 1993).

As originally proposed (Crutchfield & Kaneko 1987, 1988), the dripping handrail was an example of just such a map lattice, consisting of circle maps\(^4\) coupled linearly to nearest neighbors, one such oscillator at each point of a one-dimensional spatial lattice. We have modified the original model by using a slightly different form for the coupling (Scarle et al. 1993; Imamura et al. 1993; Steiman-Cameron et al. 1993).

We do not claim that the macroscopic discreteness assumed in our model is actually present in astrophysical systems. We know of no observational tests to distinguish between the various degrees of discreteness, although in principle such tests are possible—say, using differences in the predictions of continuous and discrete versions. In the parameter ranges we take as most appropriate for astrophysical accretion systems, such differences appear to be insignificant.

We take the spatial extent of the system to be in one dimension. Two- and three-dimensional analogs can be readily considered ("dripping pie tins," etc.) at the expense of some added complexity. To represent the geometry of the circular inner edge of an accretion disk, the spatial coordinate is wrapped around (periodic boundary conditions). This one-dimensional space is discretized in to cells, labeled by index \(i\). Similarly, time is discretized and represented by index \(n\). The matter density is taken to be a continuous variable (see the discussion of the lattice gas model in the Appendix for the case of discrete density); its value at time \(n\) and averaged over cell \(i\) is written \(\rho_i\).

For simplicity, we first discuss isolated cells by considering the case of no diffusive coupling. Each of the cells independently obeys the dynamical law

\[
\rho_{n+1} = (s\rho_n + \omega) \mod 1, \quad (1)
\]

where \(\omega\) and \(s\) are constants, and the index \(i\) is suppressed since we are here considering only isolated cells. This relation is called a linear circle map in the dynamical systems literature and is illustrated in Figure 1.

Whether or not this map generates chaotic behavior is simply determined by the slope \(s\). If \(s > 1\), the map is chaotic; if \(s < 1\), it is not. There is a fixed point in both cases: an unstable one at

\[
\rho = \frac{\omega - 1}{1 - s}, \quad s > 1 \quad (2)
\]

or an attracting one at

\[
\rho = \frac{\omega}{1 - s}, \quad s < 1. \quad (3)
\]

The circle map in the DHR has the special intermediate value \(s = 1\), dictated by conservation of matter:

\[
\rho_{n+1} = (\rho_n + \omega) \mod 1. \quad (4)
\]

For this case, if \(\omega > 0\), there are no fixed points, and the evolution is periodic with period \((1/\omega)\) if \(\omega\) is rational and quasi-periodic\(^5\) (and ergodic) if not.

Note two important points. First, there is a natural frequency, \(\omega_0\), associated with the local dynamics of the handrail. The corresponding period \(P = (1/\omega)\) is just the time it would take to fill an initially empty isolated cell—i.e., with no diffusion in or out—to the threshold value at the given accretion rate. This is the dominant timescale throughout the range of the physical parameters.

Second, because \(s = 1\), no chaos is generated "locally," i.e., by the circle maps themselves. We will see below that there are other means for generating chaotic behavior in the global (diffusive) dynamics, despite this lack of locally generated chaos.

We now add the coupling between neighboring locations (cells) provided by diffusion. The most general form of map

\[\text{\footnote{A map is a formula updating a variable at the new time step, } X_{n+1} \text{ in terms of the previous value, } X_n \text{ and possibly values at even earlier times. The circle map is the simple case in eq. (1) below. Note that the circle map has nothing at all to do with the circular geometry we are adopting for the rail!}}\]

\[\text{\footnote{The term quasi-periodic is here used in the nonlinear dynamics sense for a composite motion consisting of oscillations at two or more incommensurate frequencies.}}\]
Fig. 1.—Linear circle map $\rho_{n+1} = (\rho_n + \omega) \mod 1$ for various values of the slope $s$ and intercept $\omega$. The arrow in each panel indicates an initial condition with the dotted line showing subsequent iterations of the map. In the top left-hand panel, $s < 1$, and the attractor is a fixed point at $\rho = \omega/(1 - s)$. In the top right-hand panel, $s > 1$, and the dynamics is chaotic with an unstable fixed point at $\rho = [(\omega - 1)/(1 - s)]$. In the lower two panels, $s = 1$, and the intercept $\omega$ can be interpreted as a constant value added at each iteration. In these cases, there is no fixed point, and because $s = 1$ the behavior cannot be chaotic; hence, it is periodic or quasi-periodic. When $\omega$ is rational, the orbit will repeat, and if it is irrational, the orbit will not; i.e., the behavior is ergodic with the orbit eventually coming arbitrarily close to every point in the interval. Note, however, that in either of these cases the iterates increase for $1/\omega$ steps and are then folded back into the interval. Thus, a power spectrum of the time series of iterates will have a strong component at frequency $1/\omega$.

The lattice model considered here specifies the density at a particular site (cell) at the new time step $n+1$, as a function of the density at the sites located in a neighborhood of size $2R + 1$, at the previous time step $n$; that is,

$$\rho_{n+1}^{i} = f_P(\rho_n^{i-R}, \rho_n^{i-R+1}, \ldots, \rho_n^{i}, \ldots, \rho_n^{i+R-1}, \rho_n^{i+R}). \quad (5)$$

The function $f_P$, where $P$ is a set of parameters, thus completely specifies the dynamical aspects of the model. Note that this dynamics is deterministic and that we are not invoking any kind of intrinsic randomness or postulating the presence of noise in the form of observational errors. The meaning of the statement that the interactions are local is that the radius $R$ of the system is small, or certainly finite. If $R = 1$, only nearest neighbors are coupled, and for $R = 0$, there is no coupling.

The map lattice implementation of the dripping handrail is the following special case ($R = 1$):

$$\rho_{n+1} = [\rho_n + \Gamma(\rho_{n+1} + \rho_n - 2\rho_n + \omega) \mod 1, \quad (6)$$

where $\Gamma$ represents diffusion and $\omega$ represents accretion. Any value over threshold is wrapped by the mod operation to below the threshold, the value of which has been scaled to 1. (An alternative formulation drives values above threshold to zero, rather than subtracting the threshold; in practice this difference has little effect.) Because of the thresholding operation, the DHR system is dissipative. For a circular rail, density is defined on a one-dimensional periodic lattice of length $N_L$, and we accordingly identify the cell at $n = N_L + 1$ with that at $n = 1$.

The specific form for the coupling in this equation is chosen to represent diffusion; i.e., the transfer of matter between neighboring cells is driven by density gradients. Hence, the DHR can be referred to as a set of diffusively coupled circle maps. In particular, the term $\Gamma(\rho_{n+1} + \rho_n - 2\rho_n - 2\rho_n)$ is a discrete analog of the Laplacian (second difference) in the standard diffusion partial differential equation (PDE), and accordingly $\Gamma$ is a dimensionless analogue of the diffusion coefficient. The rate at which matter
accretes into the system is given by the parameter \( \omega \). Thus, the three postulated physical elements, accretion, diffusion and the operation of a density threshold, are implemented in equation (6).

Two opposite limits of the ratio of the accretion and diffusion parameters, \( \omega \) and \( \Gamma \), help to elucidate the behavior of the map lattice DHR, namely large diffusion \( ([\Gamma/\omega] \to \infty) \) and negligible diffusion \( ([\Gamma/\omega] \to 0) \). Before describing the behavior in these limits, we note that if, at some time \( n \), the conditions

\[
1 - \omega < \rho_i^n < 1
\]

hold at all locations \( i \) on the rail, then all sites will go over the threshold together. Physically, this equation just means that the density is everywhere close enough to threshold that the accretion in just one time step will take the density over threshold. In this case, diffusion will act to decrease the local site value differences further at an exponential rate for all future times \( m > n \). When diffusion dominates, all values on the rail attain the condition in equation (7) before reaching threshold; thenceforward, the time evolution is a completely coupled, lockstep motion and, as time goes on, approaches the solution corresponding to the one-dimensional map:

\[
\rho_i^t = (\omega n + c) \mod 1 ,
\]

(8)

at each site, where \( c \) is some constant that depends on the time at which the site values "lock" but is independent of site index \( i \). Note that in this case, as in all diffusive processes, information about the initial condition is lost.

In the other extreme, when accretion dominates, significant diffusion of material between cells does not have time to occur, so all cells almost independently fill and dump with period \( (1/\omega) \). Their phase is determined by their initial value, i.e.,

\[
\rho_i^t = (\omega n + \rho_i^0) \mod 1 ,
\]

(9)

at each site. So in both of these extremes, the evolution is periodic with fundamental period, \( P = (1/\omega) \), at which the cells fill. Higher temporal harmonics occur in the accretion dominated regime; these are damped out in the diffusion-dominated regime, as are local spatial structures.

For intermediate cases, there is a spatial scale over which density variations are effectively smoothed—roughly the diffusion distance for one period. [Here and elsewhere, when we refer to this system's period, we mean simply \( (1/\omega) \) even if the evolution is not strictly periodic.] In general, the behavior of the map lattice is more complicated than either the "lockstep" (diffusion-dominant) or independent (accretion-dominant) behavior. In the intermediate situation, competition between the smoothing of diffusion and the roughening effects of accretion with thresholding produces more complex and interesting asymptotic and transient behaviors. However, the simple relaxation oscillations that underlie even this more complex behavior explain various aspects of the handrail's evolution, in particular the central frequency of a broad QPO-type feature in the power spectrum of time series for numerically simulated map lattices.

As noted above, the DHR has no local generation of chaos. Unless otherwise stated, when we use the term chaos, we mean that nearby trajectories diverge from each other exponentially, and thus there is a positive Lyapunov exponent (Benettin, Galgani, & Strelcyn 1976; Schuster 1988).

We demonstrate in an appendix that the coupling between the circle maps in our case cannot generate asymptotic chaos either.

Dissipative systems, such as the DHR, are asymptotically governed by an attractor. In view of the above result, we conjecture that there is no asymptotic chaos possible, i.e., the most complicated type of asymptotic behavior we can expect is a limit cycle or motion on a torus (the quasi periodicity of the dynamics). We reiterate, however, that in principle the discontinuity in the map could introduce asymptotic chaos; for the conjecture to become a theorem, this loophole would have to be dealt with. In fact, we conjecture that all attractors in our case are limit cycles, as is the case for any damped, driven oscillator. Fixed points are excluded by the constant accretion term—except for the trivial cases in which the accretion rate is an integer times the threshold.

2.2. The Dripping Handrail as a Partial Differential Equation

In this section we briefly discuss the solutions of the standard partial differential equation (PDE) describing diffusion as an aid in understanding the behavior of the DHR. Consider the linear PDE for diffusion with accretion (but first without thresholding!)

\[
\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2} + A ,
\]

(10)

where \( D \) is the diffusion coefficient (related to \( \Gamma \) above) and \( A \) is the accretion rate (related to \( \omega \) above), and its solution

\[
\rho_0(x, t) = At + \sum_k \rho_{k0} e^{-dk^2 t} e^{-ikx} .
\]

(11)

The coefficients \( \rho_{k0} \) are determined by the initial value \( \rho(x, 0) \). The time dependence of this solution clearly exhibits two phenomena: (1) secular (linear) growth as material is accreted, and (2) exponential damping of the spatial Fourier modes present in the initial state, by a factor of \( e^{-dk^2} \) per unit time. Note the important point that the strength of the damping of any mode depends on the square of the wave-number \( k \) of that mode. In particular, the higher the spatial frequency, the stronger the damping due to diffusion.

Now consider the real problem, namely with the density threshold operating. Equation (11) above is a solution to the real problem, so long as

\[
0 \leq \rho(x, t) < 1 ,
\]

(12)

at all \( x \) and \( t \). Furthermore, substitution into the equation itself verifies that

\[
\rho(x, t) = \left( At + \sum_k \rho_{k0} e^{-dk^2 t} e^{-ikx} \right) \mod 1
\]

(13)

is a solution to the full problem given by equation (10) subject to the side condition in equation (12). Discontinuities at those places at which the side condition operates make the derivatives undefined.

Any time the density exceeds the threshold, the local value changes, and we effectively have a new initial condition—which has discontinuities. The combination of accretion and thresholding therefore serves to increase differences in the density along the rail. Specifically, to reiterate, since the local spatial slope of \( \rho(x, t) \) determines the
local difference, i.e.,
\[ |\delta \rho| \propto |\frac{\partial \rho}{\partial x}| \propto |k\rho|, \]
accretion with thresholding enhances high spatial frequencies in contrast to diffusion, which strongly damps them. Another interesting point is that, as opposed to the discrete time case, there is no longer a tube of radius \( \omega \) about the lockstep periodic solution, here represented by steady growth and reinjection along the diagonal of the (uncountably) infinite dimensional hypercube. In this case, even infinitesimal differences in density along the rail are enhanced by accretion and thresholding.

2.3. The Dripping Handrail as a Cellular Automaton

This section outlines an implementation of the DHR based on a cellular automaton (CA), in which not only are space and time discrete but in which also the density can take on only a finite number of discrete values. The evolution is determined by rules that fix the density in a given cell at the next time step as a function of its current density and those of its nearest neighbors. This simple construct, and the related lattice gas, have yielded computational procedures of considerable power for the solution of fluid problems (Hasslacher 1989; Doolen et al. 1990), including a number of cases of astronomical interest (Perdang & Lejeune 1993).

We have experimented with a CA rule that was chosen to implement the three elements of the processes defining the DHR (diffusion, accretion, and a threshold). Below is a listing of the MatLab (trademark of the MathWorks, Inc.) script that implements the DHR CA for an arbitrary number of states. While it is strictly speaking not pseudo-code, with a few explanations of some basic MatLab constructs, the reader should be able to use this script to write an analogous program in any language. The generation of an index array \( t \) is accomplished by a statement of the form 
\[ t = 1:n, \]
which generates the array \((1, 2, 3, \ldots, n)\). Adding a scalar to an array increases every element of the array by the value of the scalar, so \( t \)=1:3\) generates the array \((2, 3, 4, \ldots, n+1)\). Setting one element of an array is accomplished with a statement of the form \( t(n) = 1 \), which when executed after the above command generates the array (still of length \( n \)) \((2, 3, 4, \ldots, n, 1)\). An expression of the form
\[ t > c, \]
where \( t \) is an array and \( c \) is a scalar, generates an array that is zero for indices \( i \) for which \( t(i) < c \) and unity for those for which the inequality \( t(i) > c \) is satisfied. Finally, since \( * \) is reserved for matrix multiplication, \( a*b \) means to multiply the elements of two arrays \( a \) and \( b \) (which must be of the same size) together, term by term. Comments following the \% symbol are ignored.

% DRIPPING HANDRAIL CELLULAR AUTOMATON
%
num_cells=64; % number of spatial cells
max_time=512; % number of time steps
max_mass=10; % number of states
threshold=max_mass; % threshold level
accretion=1; % accretion rate
**MAKE SHIFT OPERATORS:
% x(tpl)=Shift-to-right(x)
% x(tml)=Shift-to-left(x)
%******************************************************************************
tpl=(1:num_cells)+1;
tml=(1:num_cells)-1;
% Wraparound (periodic boundary condition):
tpl(num_cells)=1;
tml(1)=num_cells;
%******************************************************************************
% INITIAL CONFIGURATION OF RAIL
%x=fixed(rand(1, num_cells)*max_mass);
%******************************************************************************
% EVOLVE SYSTEM FORWARD IN TIME
%******************************************************************************
for time=1:max_time
  excess=x-x(tpl); % EXCESS OVER RIGHT-HAND NEIGHBOR
  f_right=excess.*(excess>0); %
  non-negative part of this excess
  excess=x-x(tml); % EXCESS OVER LEFT-HAND NEIGHBOR
  f_left=excess.*(excess>0); %
  non-negative part of this excess
  f_net=f_right-f_left;
  move_right=f_net.*(f_net>0);
  move_left=f_net.*(f_net<0);
  x=x-move_right+move_left; % MASS LEAVES CURRENT CELL...
  x(tpl)=x(tpl)+move_right; % AND ENTERS ITS RIGHT-HAND NEIGHBOR
  x(tml)=x(tml)-move_left; % OR ENTERS ITS LEFT-HAND NEIGHBOR
  x=x+accretion; % CONSTANT ACCRETION
  x=x.*((x<threshold)); % KEEP ONLY WHAT IS BELOW THRESHOLD
  rail(time)=sum(x); % STORE TOTAL CONTENTS OF RAIL
end

The diffusion is based on a pseudoforce—defined as the amount by which a cell's contents exceeds that of its neighbors. (What enters a cell is determined implicitly, based on what leaves its neighbors; the pseudoforce is not explicitly used to determine the mass entering a cell.) The amount of mass that leaves a cell is equal to the magnitude of the pseudoforce; the direction to the neighbor that gets this mass is determined by the sign of the pseudoforce. There is no diffusion of matter from an empty cell. This system is equivalent to a deterministic truth table that gives the evolution of a cell as a function of the occupation of its four nearest neighbors.

Note that this CA is different from the standard way of implementing diffusion in lattice gas systems (Bohgosian & Levermore 1987) in that we use only the mass density as the state variable—velocity is not explicitly used as a variable at all. We have run tests of just the diffusion part of our CA to see how well it represents the physics of diffusion. One test was to set up an initial state consisting of a block of gas embedded in a vacuum; the result is that the gas expands into the vacuum at a rate proportional to the square root of time (as expected for diffusion). However, our CA is not a CA version of Burgers's equation, as is that of (Bohgosian & Levermore 1987). This is because our CA ignores the velocity of the fluid that is one of the dependent variables in Burgers's equation.
The full DHR CA produces time series that are similar to that of the ML version of the DHR, in that there is a QPO feature in the power spectrum, as well as some very low frequency noise. This can be best seen in Figure 3, which compares the power spectra of the time series produced by three different implementations of the dripping handrail. An another set of rules, for the case of a three-state system, consists of three steps performed in sequence:

1. **Diffusion**.—Each nonempty cell C gives one unit to an emptier neighbor, unless C's other neighbor is also emptier than C. If C has two units, it gives them to an empty neighbor if and only if C's other neighbor also has two units.

2. **Accretion**.—Add one unit to each cell at each time step.

3. **Threshold**.—Empty any cell with more than two units.

This simple set of deterministic rules yields dynamical evolution that has essentially the same character as that described by equation (6), including QPOs and VLFN. Other spatially extended diffusive systems, such as reaction-diffusion systems, have similar behavior (Tamayo & Hartman 1989; see their Figs. 10–12, which show QPOs and VLFN). Boghosian & Levermore (1987) proved that a similar diffusion cellular automaton solves the standard PDE formulation of nonlinear diffusion. Their numerical results agree with analytic solutions, including the formation and evolution of shock discontinuities. Our rules are considerably simpler than theirs, as we use neither particle velocities nor random variables. The conclusion is that the production of QPOs and VLFN is a robust property of the physics and is not due to the discrete/continuous character of the various representations or careful selection of parameters.

3. **DHR PHENOMENOLOGY**

This section deals with the nature of the behavior of the handrail by reporting on the results of extensive numerical computations of the DHR evolution, mostly with the map lattice (ML) version. Although fine details, such as the nature of the individual drops, may be of interest physically—for astronomical applications, only the total behavior of the rain can be observed (because accretion disks cannot be resolved with current observations). After discussing connections with observations, we study the behavior of dripping handrail simulations and show that its simple physical mechanisms can account for the types of phenomenology seen in data from astrophysical systems thought to be driven by accretion. We give detailed results of numerical simulations of the dripping handrail for four sets of parameter values, arranged in terms of an increasing ratio of diffusion to accretion. We characterize the evolution of the system by computing Lyapunov exponents and by examining the time series corresponding to the observable luminosity (indirectly through the unobservable matter density). In turn, the time series are studied by examining their power spectra and wavelet spectra ("scaleograms") for various values of the model physical parameters.

### 3.1. **Connection with Observations**

One of the main issues in examining the behavior of the handrail is the question of which properties should be compared with observational data. In the following, we simply assume that the effective luminosity at any given time is proportional to the total mass on the rail, which is reasonable if, e.g., we are considering the intensity of X-ray emission from the hot \((\approx 10^6 \text{ K})\) thermalized inner edge of an accretion disk. As such, we will study the time series of the total amount of material at each time step in simulations of the handrail. (In future work, we will develop more realistic procedures for computing the luminosity of the DHR model.) For comparison, we might examine the time series of the amount of material that falls off the rail at each time step, but as this is simply the derivative of the total mass time series, we will not consider it here. For more discussion on the comparison between these two different choices of observable, see Steiman-Cameron et al. (1993). Power and wavelet spectra of the corresponding time series serve as the main diagnostics. In addition, we will study pictures of the time development of the mass distribution because they are illustrative of interesting physical relationships. And, as a heuristic means of demonstrating the transient nature of the rail's chaotic behavior, we determine the largest Lyapunov exponent and study it as a function of the number of rail sites, \(N_L\).

#### 3.2. **Characteristics of DHR Light Curves**

In order to exhibit the behavior of the system as a function of its physical parameters is an efficient way, we have adopted the scheme in Figure 2, which shows sequence of 50 parameter values along a circular arc through \(D, A\) parameter space. The reason for choosing this particular path is that it is along the direction of maximum change in the ratio of diffusion to accretion and for which we expect maximum change in system behavior. We define the angle shown in the figure as

\[
\theta = \arctan \left( \frac{\Gamma}{\omega} \right),
\]

where \(\Gamma\) and \(\omega\) are the diffusion and accretion parameters defined above. Figure 3 (Plate 5) shows the dependence of the
power spectrum on the parameter representing the ratio of diffusion to accretion. The three panels show, respectively, the results for the three different DHR implementations: the PDE model in equation (13); those from numerical simulations of the ML, using equation (6); and the last for the cellular automaton version, using the MatLab code in § 2.3 above. Each of the 512 spectra refers to a single random initial condition, different for each parameter value. We examined the resulting plot made by averaging over a number of randomized initial conditions and for the case in which the initial conditions were random as a function of position along the rail but were the same for all parameter values. The results were not significantly different from those shown in Figure 3.

This figure demonstrates several of the characteristics of the handrail discussed above. Concentrate on the central panel, which is for the map lattice—but much of the structure shown there is echoed in the other two panels. The initial spectra, for low values of the ratio of diffusion to accretion, show sharp peaks at the fundamental frequency \( \omega \) and its first few harmonics (with the others beyond the Nyquist frequency). For the asymptotic solution corresponding to lockstep motion, the time series is a sawtooth that has a power spectrum with the \( n \)th harmonic having a coefficient proportional to \( n^{-2} \). The spectra at low diffusion-to-accretion ratio, averaged over 20 random initial conditions per parameter setting, have coefficients that follow a similar, but slightly larger, decrease with \( n \).

Figure 4 shows the power in the low bin (the fundamental frequency, not zero frequency) of the power spectrum plotted as a function of parameter setting. In this plot, we explicitly see the onset of very low frequency noise (VLFN) behavior at parameter number 32 out of 50, that is \( \theta_{32} = (32/50)(\pi/2) \).

In contrast, for large values of the ratio, there is no power or observable structure at high frequencies as expected for the diffusion dominated regime. The rise in power at low frequency in this regime is apparent in both Figure 3 and Figure 4. There is essentially no low-frequency structure until the ridge at parameter setting \( \theta_{32} \approx 0.98559 \), at which point the power at low frequency begins to increase steadily.

Note also from Figure 3 that the power in the peak corresponding to the fundamental frequency \( \omega \) decreases from its initial value and vanishes before the onset of VLFN-type behavior. Right before this onset, again approximately at parameter setting \( \theta_{32} \), a broad band that we identify as the quasi-periodic oscillation (QPO) appears. This peak broadens with increasing ratio of diffusion to accretion, as do the other harmonics.

Both the QPOs and VLFN are behaviors that span a large range of both space and timescales. We will use wavelet spectra to demonstrate this, but we first discuss a heuristic measure of scale invariance. Accretion with thresholding excites events, i.e., drops, on the smallest scales, and diffusion, via smoothing, distributes the power over larger scales. As a means of providing support for this picture we show, in Figure 5 a three-dimensional plot of event histograms accumulated for the same set of 50 parameters settings as used in the power spectra plots. By an even is meant any loss of mass from one step in the evolution of the handrail to the next. This can include the combination of any number of disconnected sites exceeding threshold. Hence, events do not directly correspond to drops in the sense of contiguous regions of the rail, but they are the analog of drops appropriate to situations in which only summations over the rail—and not individual cells—can be observed.

To obtain the histograms, 20 simulations of 32,768 time steps for handrails of 251 sites were done at each parameter setting. If there was a net loss between any two iterations, the histogram value associated with the event (amount lost) of that size was increased by one. The histograms were accumulated over all the runs for a given parameter. One indication of scaling behavior is evidence of a power-law relation between numbers of events and their size, \( N(s) \approx Cs^\gamma \) (where \( \gamma \) is some constant) i.e., an approximately linear relation in a log-log plot of \( N(s) \). For the histograms at low diffusion-to-accretion ratio, we see a "knee" in the plot that indicates a characteristic scale as well as a deviation from linear behavior. For the larger values, at and beyond the onset of the VLFN and QPO behavior, the plots appear linear over a range of drop sizes.

We next turn to a detailed study of the behavior at four particular parameter settings from small to large ratio of diffusion to accretion parameter. For these, Figure 6 shows a plot of the density along the rail at a particular time, and

\[ \tau, \theta_i \]

Fig. 4.—Power spectrum at the lowest frequency (the first harmonic), as a function of the parameter defined in Fig. 2. This figure demonstrates how the random part of the time series (that is, the very low frequency noise component) grows with the importance of diffusion relative to accretion (as measured by \( \theta \)).

\[ \log(# \text{ of Drops}) \]

Fig. 5.—Histogram of "drop size," defined in the text, as a function of \( \theta \).
TRANSIENT CHAOS IN ACCRETION SYSTEMS

Fig. 6.—Density of matter as a function of position on the rail (at arbitrary times in the evolution of the systems) to illustrate the degree of irregularity. The four values of $\theta$ indicated in the figure (and used for the subsequent few figures) were chosen to span the interesting range.

Figure 7 shows a spacetime plot of the evolution of the handrail from a particular initial condition. These figures were intended to provide some visual intuition about the space scale and timescale of relevant structures, i.e., blobs, on the rail.

For the four parameter settings we show the time series in Figure 8, two periods of a folded pulse in Figure 9 [where by period, here we mean simply $\langle 1/\omega \rangle$], the power spectra in Figure 10, the wavelet scalegrams in Figure 11, drop histograms in Figure 12, and Lyapunov exponents in Figure 13.
The folded pulses, power spectra, and wavelet scalegrams are averaged over 20 runs for handrails of 37 sites, and 16,384 iterations, beginning with random initial conditions. The time series in the plot is a segment of a longer time series, beginning with iteration 12,000 and ending with iteration 12,599, for one of these runs.

Figure 9 was constructed to allow comparison with the folded light curves that observers frequently assemble from

Fig. 9.—Pulse shapes derived for the four systems described in Fig. 6, obtained by folding the time series at the period corresponding to the fundamental filling rate defined in the text. For X-ray pulsars, this is the procedure used to process observational time series. The shapes of the folded pulses bear a resemblance to the pulses thus derived for some X-ray pulsars, suggesting that some of the "pulsing" in some of these objects may be due to periodic modes in an accretion disk, not to a beam attached to a rotating neutron star.
time series data for sources in which they have identified a period. The resulting “pulse shapes” for the periodic modes characteristic of larger accretion rates bear some resemblance to the similarly determined shapes for some X-ray pulsars. These curves tend to be quite distinctive, non-sinusoidal, and characterized by plateaus (see, e.g., Trümper et al. 1986; Parmar et al. 1989b; Parmar, White, & Stella 1989a, especially Fig. 1). We speculate that some of the
pulsing observed in some of these systems may be such periodic-mode accretion and not be due to pure rotation as in the standard pulsar model.

The largest Lyapunov exponent is calculated for the evolution of the $N$-site map lattice as an $N$-dimensional dynamical system at the four parameter settings using the method described in (Benettin et al. 1976). At each of the four values of $\theta$, we do this for handrails with the number of sites ranging from three through seven. For small numbers of sites, the rail can decay to its asymptotic limit cycle in a

![Graphs showing Lyapunov exponents and histograms of drop sizes](https://via.placeholder.com/150)

**Fig. 12.**—Lyapunov exponents, as a function of time—expressed in time steps, or “iterations,” derived for the four systems described in Fig. 6

**Fig. 13.**—Histograms for drop sizes, as defined in the text, for the four systems described in Fig. 6
relatively short time, which is reflected in the value of zero obtained for the Lyapunov exponent obtained in these cases. As the number of sites increases, the decay time grows hyperexponentially, and this is reflected in the nonzero values obtained for the Lyapunov exponents in these cases.

An examination of Figure 12 (the Lyapunov exponents calculated as a function of time) shows that in fact the value of the exponent reaches a plateau and remains at that value for a period of time that is a function of the number of sites on the rail. For low-dimensional systems, i.e., those with a small number of degrees of freedom, this is an indication of chaos. In the present context, it provides evidence for the behavior we are describing as transient chaos, i.e., apparently chaotic behavior in the transient behavior of a system. In this case in fact, the system is asymptotically periodic, i.e., asymptotically nonchaotic.

4. CONCLUSIONS

We have provided a detailed introduction to the dripping handrail as a model that describes the behavior of a wide range of astrophysical systems in which accretion is thought to be a dominant component of energy production. Behaviors observed in actual systems—periodicities, very low frequency noise (VLFN), and quasi-periodic oscillations (QPOs)—have been obtained from simulated time series of the total mass on the rail at various parameter settings.

The model provides physically motivated explanations for these behaviors in a simple and unified framework; i.e., all the phenomenology is a result of the properties of diffusion and accretion with thresholding on the inner edge of an accretion disk. We argue that the observed behavior of a variety of accretion systems can be associated with the transient behavior of something like a DHR system. Any temporal or spatial inhomogeneities associated with the natural driving mechanism for the system, i.e., the accretion mechanism, tends to “reset” the system long before it has had time to decay to its asymptotic behavior. Furthermore, using standard measures of chaotic behavior, e.g., Lyapunov exponents, in certain parameter regimes, time series produced by the DHR appear to be chaotic. This is the case despite the fact that the asymptotic behavior of the DHR is explicitly not chaotic, as we demonstrated. This supports the contention that the typical behavior of such systems is to be associated with transient chaos.

We plan to investigate the predictions that this model makes for things like the correlation between accretion rate and the QPO frequency—or more generally between accretion rate and the structure of the power spectrum of the fluctuations of the system. In addition, by adding more physics to the DHR, we hope to develop a model that is less artificial and able to be compared in more detail to existing observations and those that will be obtained by the new generation of X-ray telescopes but that retains its straightforward phenomenological interpretation.

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APPENDIX A

RELATIONSHIP BETWEEN PDE, MAP LATTICE, AND CA MODELS

In this appendix, we compare the map lattice, partial differential equation, and cellular automaton version of the DHR. We use the perspective of dimensional analysis to provide a unified approach.

We could start with a completely discrete or a completely continuous model in terms of time, space, and field variable and obtain all variants by taking limits or discretizing respectively. For the purposes of our discussion, we choose to begin by discussing the fully discrete model first, but in a fundamental sense this an arbitrary choice. To this end, consider a simple, completely discrete (i.e., in space, time, and field variable) system that models the above competing effects

\[ \rho_{n+1}^i = \left( \rho_n^i + \sum_{j=i-R}^{i+R} c_j \rho_n^j + \Omega \right) \mod T. \]  

(A1)

Here, the field variable \( \rho_n^i \) is an integer number of “unit” mass values \( \Delta \rho \), with space index \( i \) and time index \( n \). For each iteration of unit time step \( \Delta t \), an integer number of unit masses, \( \Omega = A \Delta t \), is added at each site at which \( A \) is the “physical” accretion rate per unit time. In general, the amount of mass added would be a function of time and space indices, but we first want to explore the simplest version of the model. The summation over a local neighborhood of sites of size \( 2R + 1 \) is a general smoothing that models diffusion. Finally, \( T \) is an integer number of mass units \( \Delta \rho \) characterizing the maximum density on the rail, the mod operator modeling the dumping of material when the local density exceeds \( T \). In general, the coefficients \( c_j \) must be chosen as integers so that the sum and hence the resulting site value is an integer. This requirement in

\[ \text{We will address more sophisticated models in future work. For instance, a next step would be to examine the properties of a one-dimensional lattice gas version of Burgers's equation, which is a diffusion equation with a bulk velocity term. This would allow us to address more realistic models that involve shocks and transfer of angular momentum. These phenomena might be particularly relevant in two-dimensional thin disk models because in these models the fact that the Kepler velocities can be much greater than the sound velocity leads to models with a large shock-induced viscosity thought necessary for transporting angular momentum outward in the disk. In this paper, we are specifically trying to see what we can learn about the simplest possible model with interesting dynamics.} \]
fact dictates that an exact local cellular automaton model of diffusion (i.e., a discrete model for which the local site evolution is completely described by the values on a bounded neighborhood of the given site) is not possible.\(^7\) Reasonable cellular models, which are not completely local, are, however, possible as was demonstrated in § 2.3.

We next consider a more specific form of equation (A1) with a local neighborhood for which \(R = 1\), and we choose coefficients to mimic the continuous diffusion equation

\[
\rho_{n+1}^i = \left[ \rho_n^i + \Gamma (\rho_{n+1}^{i+1} + \rho_{n-1}^{i-1} - 2\rho_n^i) + \Omega \right] \mod T .
\]

\((A2)\)

Here \(\Gamma = [D \Delta t/(\Delta x)^2]\) is a dimensionless measure of the diffusion strength, and \(D\) is the “physical” diffusion per unit time over unit distance squared. This is an eight-parameter model, i.e.,

\[
\rho_n^i = \rho_n^i(\Delta \rho, \Delta x, \Delta t, \Gamma, \Omega, T, L, t_{\text{final}}) ,
\]

\((A3)\)

where \(L\) is the size of the rail and \(t_{\text{final}}\) is the relevant timescale over which the system is to be studied. Note that we can express the threshold, the size of the rail, and the observation time in terms of dimensionless measures of size \(N_L = T/\Delta \rho, N_L = T/\Delta x,\) i.e., the number of discrete sites on the rail, and \(N_t = t_{\text{final}}/\Delta t,\) i.e., the number of discrete time steps. We can obviously eliminate parameters by fixing them; in the cases of \(\Delta \rho, \Delta x,\) and \(\Delta t,\) we can eliminate them by taking appropriate limits when these units are small compared to the relevant quantities \(T, L,\) and \(t_{\text{final}},\) respectively. For example, to generate the map lattice dripping handrail model, we consider the case for which \(\Delta \rho < \Omega,\) i.e., \(\Delta \rho \to 0.\) To obtain a dimensionless model, we divide both sides of equation (A2) by \(T\) and obtain

\[
\rho_{n+1}^i = \left[ \rho_n^i + \Gamma (\rho_{n+1}^{i+1} + \rho_{n-1}^{i-1} - 2\rho_n^i) + \omega \right] \mod 1 ,
\]

\((A4)\)

where \(\omega = \Omega/T\) and \(\rho_n^i \to \rho_n^i/T\) is now a unitless, real-valued quantity between 0 and 1. This is the basic model we have been working with.

The map lattice model appears to be a six-parameter model

\[
\rho_n^i = \rho_n^i(\Delta x, \Delta t, \Gamma, \Omega, L, t_{\text{final}}) ,
\]

\((A5)\)

but by Buckingham’s pi theorem and consideration of the fundamental units of space and time (we eliminated the fundamental mass unit in the previous step), it turns out to be a four-parameter model. In fact, in certain limits (large number of site values and time steps), the model can be considered equivalent to a two-parameter, continuous PDE model. We can obtain the form of this model directly by steps similar to those above. First, consider the limit of a large number of time steps, i.e., \(\Delta t \ll t_{\text{final}}\) but finite \(\Delta x.\) In this case, we divide both sides of equation (A4) by \(\Delta t\) and take the appropriate limit

\[
\lim_{\Delta t \to \infty} \left( \frac{\rho_{n+1}^i - \rho_n^i}{\Delta t} \right) = \lim_{\Delta t \to \infty} \left[ \frac{d(\rho_{n+1}^{i+1} + \rho_{n-1}^{i-1} - 2\rho_n^i) + a}{\Delta t} \right] \mod 1
\]

\((A6)\)

with \(d = \Gamma/\Delta t = D/(\Delta x)^2\) and \(a = \omega/\Delta t = \Omega/T/\Delta t = A/T.\) The quantity \(a\) is a dimensionless measure of what percentage of the threshold value is accreted per site per unit time. This version of the model is a coupled set of ordinary differential equations. Finally, if we consider the limit of a dynamical system with an infinite number of degrees of freedom \(N_L,\) i.e., a PDE model with \(L \gg \Delta x,\) we obtain

\[
\lim_{N_L \to \infty, \Delta x \to 0} \left( \frac{d\rho^i}{dt} \right) = \lim_{N_L \to \infty, \Delta x \to 0} \left[ \frac{D (\rho^{i+1} + \rho^{i-1} - 2\rho^i)}{(\Delta x)^2} + a \right] \mod 1
\]

\((A8)\)

This is now a two-parameter model of an accreting (with threshold) diffusing handrail in which we have various choices for the two dimensionless parameters. For instance, we could use a dimensionless ratio of the timescales for diffusion around the rail and accretion to threshold \(D/aL^2,\) and a scaled ratio of position and time coordinates on the rail \(Lt/xt_{\text{final}}\) to obtain

\[
\rho = \frac{D}{aL^2} \cdot \frac{Lt}{xt_{\text{final}}}
\]

\((A10)\)

An important point is that the map lattice version of the handrail is the forward time-centered space (FTCS) finite difference approximation to the partial differential equation version of the handrail. In fact, by the von Neuman stability criterion (Press et al. 1988), if \(\Gamma \leq \frac{1}{2}\), then the solutions of the map lattice model asymptotically approximate the solutions of the PDE model as discussed above. As has been noted elsewhere (Press et al. 1988), the physical interpretation of this condition is that for the lattice map version to be considered a stable approximation of the PDE model, the largest allowed time step \(\Delta t\) is given by the diffusion time across a cell of width \(\Delta x.\) We have restricted the parameters in the map lattice model to obey this criterion; hence, any information we obtain about the asymptotic solutions of the PDE model is relevant to the analysis of the dripping handrail. In this case, we can recast the above relation in terms of the parameters of the map lattice model by using the above

\(^7\) This is in turn related to the infinite phase velocity associated with the continuous diffusion equation.
where
\[ 0 \leq i \leq N_L - 1 \] (A12)
is the site value and
\[ 0 \leq n \leq N_L - 1 \] (A13)
is the time step. With \( N_L n / iN_L \) serving as a combination of the spacetime coordinates on the rail, it would appear that we could examine the various generic properties of the rail simply in terms of the single parameter \( \Gamma / \omega N_L^2 \). The problem is that for systems with large numbers of degrees of freedom, the phase space structure is generally very complicated (Crutchfield 1988; Keeler & Farmer 1986), so at any particular parameter setting, one generally expects to find a large number of attractors, i.e., fixed points, limit cycles, strange attractors, and perhaps new types, with very complicated intertwined basins of attraction that themselves have interesting dynamical structure (Crutchfield 1988). We have demonstrated this by constructing a limit cycle for the handrail and showing that from certain initial conditions, one can approach this limit cycle arbitrarily closely despite moving away asymptotically. That is, the basin of attraction in which the initial condition lies comes arbitrarily close to the limit cycle, i.e., a different attractor.

If we begin with all versions of the model in parameter regimes such that the transient and asymptotic behavior of the various versions agree, our fundamental approach has been to use whichever version we can to understand this behavior. The nonlinearity of the map lattice model might be thought of as the simplest possible, i.e., the introduction of the threshold generates a piecewise linear as opposed to a linear model. Despite this apparent simplicity, the authors are unaware of analytic solutions of any of the above models; hence, we have resorted to qualitative analysis and computer simulations for which we found the map lattice model particularly appropriate.

\section*{Appendix B}

\subsection*{DHR Lyapunov Exponents}

We will demonstrate in this appendix that the coupling between the circle maps in our case cannot generate asymptotic chaos. We note that the following analysis indicates only when chaos is introduced explicitly by the coupling. If the coupling does not introduce chaos, it might nonetheless be globally introduced by the discontinuity of the map, for example. We begin by evaluating the DHR’s Jacobian matrix, namely the \( N_L \times N_L \) matrix

\[ J = \begin{pmatrix} 1 - 2\Gamma & \Gamma & 0 & \ldots & 0 & \Gamma \\ \Gamma & 1 - 2\Gamma & \Gamma & 0 & \ldots & 0 \\ 0 & \Gamma & 1 - 2\Gamma & \Gamma & \ldots & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \Gamma & 1 - 2\Gamma & \Gamma \\ \Gamma & 0 & \ldots & 0 & \Gamma & 1 - 2\Gamma \end{pmatrix} \] (B1)

and then computing the eigenvalue spectrum of \( J \):

\[ \mu_i = 1 - 2\Gamma \left[ 1 - \cos \left( \frac{2\pi i}{N_L} \right) \right], \quad i = 0, 1, \ldots, \left\lfloor \frac{N_L}{2} \right\rfloor. \] (B2)

From the fact that

\[ \max |\mu_i| = \max \left( 1, |1 - 4\Gamma| \right), \] (B3)

then if \( \Gamma \leq \frac{1}{4} \), we have \( \max |\mu_i| = 1 \). But the necessary and sufficient condition that a discrete dynamical system, such as the DHR, be chaotic is \( \max |\mu_i| > 1 \); i.e., \( \lambda = \log(\max |\mu_i|) > 0 \).

As we will impose the condition \( \Gamma < \frac{1}{2} \) on all versions of the DHR considered in this paper, none of these cases can display asymptotic chaos due to the coupling. Significantly, this is exactly the condition for which the von Neumann stability criterion is violated, i.e., the condition for which the map lattice fails to approximate the corresponding diffusive PDE model described in the next section. For values of \( \Gamma > \frac{1}{2} \), the PDE and ML simply represent different models; it is not proper to regard one as correct, and the other an approximation that breaks down.

We also note that the eigenvalue \( \mu_0 = 1 \) is associated with the \( N_L \)-dimensional eigenvector

\[ v = (1, 1, \ldots, 1), \] (B4)

indicating that the periodic solution with all sites accreting and dumping in lockstep is neutrally stable. From the fact that \( \Gamma < \frac{1}{2} \), all eigenvalues other than \( \mu_0 \) satisfy \( |\mu_i| < 1 \). This means that at each iteration, density values between threshold
crossings are attracted to the periodic solution along directions in phase space associated with the eigenvectors corresponding to the eigenvalues \( \mu_i, i > 0 \). (Dynamicists refer to these directions as stable manifolds.)

A geometric way of thinking about this is to realize first that the phase space of the dripping handrail can be thought of as an \( N_r \)-dimensional unit hypercube with periodic boundary conditions, i.e., an \( N_r \)-torus. Lockstep evolution corresponds to motion at a constant velocity along the diagonal of the hypercube with reinsertion into the hypercube every \( [1/\omega] \) steps. The fact that points in phase space may not actually decay to the periodic solution is due to the fact that, when individual site values exceed the threshold, the phase space point is reinserted into the unit hypercube at a location that is generally far from the diagonal. This occurs regardless of the fact that the evolution over a time interval for which all site values remain below threshold, i.e., within the unit hypercube, moves the phase space point closer to the diagonal. In fact, the condition specified in equation (7) defines a tube of radius \( \omega \) about the diagonal. If diffusion is able to move a phase space point into this tube before any of the phase space coordinates exceed threshold, the asymptotic solution will be lockstep motion; otherwise, the asymptotic solution will be some more complicated limit cycle.

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Note added in proof.—With regard to the comments about high-frequency QPOs near the end of § 2.1, early XTE observations have indeed yielded a variety of kHz QPOs: in Sco X-1 (M. van der Klis, J. Swank, W. Zhang, K. Jahoda, E. Morgan, W. Lewin, B. Vaughan, & J. van Paradijs, IAU Circ. 6319 [1996]), in 4U 1728 – 34 (T. Strohmeier, W. Zhang, & J. Swank, IAU Circ. 6320 [1996]), and in 4U 1608 – 52 (J. van Paradijs et al., IAU Circ. 6336 [1996]). It appears that kHz QPOs, often exhibiting the correlation between frequency and accretion rate predicted by the DHR model, are a common feature of low-mass X-ray binaries.
Fig. 3.—Power spectra of dripping handrail (top to bottom): (a) exact solution of PDE; (b) 32 cell map lattice; (c) 32 cell, 512 state cellular automaton; logarithmic color scale for spectral power. Vertical axis: Frequency (Nyquist frequency units). Horizontal axis: \( \theta \) (arctangent of the ratio of diffusion coefficient to accretion rate). Note the similar behavior of these three very different models: As diffusion increases relative to accretion, the time series goes from periodic to quasiperiodic, with a continuous background of 1/f chaotic noise (better seen on plots of the scalegram, the wavelet analog of the power spectrum, as in Fig. 11).

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