Sec. 8.1, ex. 4: Let
\[ f_n(x) = \frac{x^n}{1 + x^n}. \]
If \(0 \leq x < 1\), then \(f_n(x) \to 0\) as \(n \to \infty\). If \(x = 1\), then \(f_n(1) = 1/2 \to 1/2\). If \(x > 1\), then \(f_n(x) = \frac{1}{x^{-n} + 1} \to 1\).

Summarizing,
\[
f(x) = \lim_{n \to \infty} f_n(x) = \begin{cases} 
0 & 0 \leq x < 1 \\
\frac{1}{2} & x = 1 \\
1 & x > 1.
\end{cases}
\]

Sec. 8.1, ex. 14: Let \(0 < b < 1\). We will show that \(f_n \to f\) uniformly on \([0, b]\). Note that \(f = 0\) on \([0, b]\). Therefore, for \(x \in [0, b]\):
\[
|f_n(x) - f(x)| = \frac{x^n}{1 + x^n} \leq x^n \leq b^n.
\]
Thus \(\|f_n - f\|_{[0,b]} \leq b^n \to 0\), as \(n \to \infty\), which means that \(f_n \to f\) uniformly on \([0, b]\).

However, we claim that convergence is not uniform on \([0, 1]\). Let \(x_n = 1 - \frac{1}{n}\). Then
\[
\|f_n - f\|_{[0,1]} \geq |f_n(x_n) - f(x_n)| = \frac{(1 - \frac{1}{n})^n}{1 + (1 - \frac{1}{n})^n} \to \frac{e^{-1}}{1 + e^{-1}} = \frac{1}{e + 1}.
\]
Therefore, \(\|f_n - f\|_{[0,1]} \not\to 0\), proving the claim.

Alternatively, since \(f_n'(x) = nx^{n-1}/(1 + x^n)^2 > 0 (x > 0)\), the function \(x \mapsto f_n(x)\) is increasing on \([0, \infty)\), so for any \(b > 0\) we obtain
\[
\|f_n - f\|_{[0,b]} = \|f_n\|_{[0,b]} = f_n(b) = \frac{b^n}{1 + b^n} = \begin{cases} 
\to 0, & \text{if } b \in (0, 1) \\
\not\to 0, & \text{if } b \geq 1,
\end{cases}
\]
proving both claims at once. \(\square\)

Sec. 8.1, ex. 19: Let \(g_n(x) = x^2 e^{-nx}\). Then \(g_n(x) \to 0\), pointwise for all \(x \geq 0\). To show that the convergence is uniform, let us find \(\|g_n\|_{[0,\infty)} = \|g_n - 0\|_{[0,\infty)}\). We have
\[
g_n'(x) = 2xe^{-nx} - x^2ne^{-nx} = 0
\]
if $x = 2/n$. Using the first or second derivative test, it is easy to see that $g_n$ achieves the global maximum at $2/n$. Therefore,

$$
\|g_n\|_{[0, \infty)} = g_n \left( \frac{2}{n} \right) = \frac{4}{e^2 n^2} \to 0,
$$
as $n \to \infty$, proving uniform convergence $g_n \to 0$ on $[0, \infty)$. □

Sec. 8.1, ex. 23: First we prove the following

**Lemma.** If $(f_n)$ is a sequence of bounded functions on a set $A$ and $f_n \to f$ uniformly on $A$, then $f$ is bounded and there exists a constant $M > 0$ such that $\|f_n\|_A \leq M$, for all $n$. That is, the uniform limit of bounded functions is bounded and the sequence itself is uniformly bounded.

**Proof.** Assume $f_n \to f$ uniformly on $A$. Then there exists $N$ such that for all $n \geq N$, $\|f - f_n\|_A < 1$. Thus

$$
\|f\|_A \leq \|f - f_N\|_A + \|f_N\|_A < 1 + \|f_N\|_A < \infty,
$$
which proves that $f$ is bounded. To show that the sequence $(f_n)$ is uniformly bounded, let $M = \max\{\|f_1\|_A, \ldots, \|f_{N-1}\|_A, 1 + \|f\|_A\}$. Then for $n \geq N$, we have

$$
\|f_n\|_A \leq \|f_n - f\|_A + \|f\|_A < 1 + \|f\|_A \leq M.
$$
The inequality is clearly true if $n < N$. Hence $\|f_n\|_A \leq M$ for all $n$. □

Now assume $f_n \to f$ and $g_n \to g$ uniformly on $A$, where $(f_n)$ and $(g_n)$ are both sequences of bounded functions. Then $\|f_n - f\|_A \to 0$ and $\|g_n - g\|_A \to 0$. By the Lemma, $\|f\|_A < \infty$ and there exists $M > 0$ such that $\|g_n\|_A \leq M$, for all $n$. Then:

$$
\|f_n g_n - fg\|_A \leq \|f_n g_n - f g_n\|_A + \|f g_n - f g\|_A
\leq \|(f_n - f) g_n\|_A + \|f(g_n - g)\|_A
\leq M \|f_n - f\|_A \|g_n\|_A + \|f\|_A \|g_n - g\|_A
\to 0,
$$
as $n \to \infty$. Therefore, $f_n g_n \to fg$ uniformly on $A$. □

**Remark:** Here we used the fact that $\|\phi \psi\|_A \leq \|\phi\|_A \|\psi\|_A$. The proof is easy: for each $x \in A$, $|\phi(x)\psi(x)| = |\phi(x)||\psi(x)| \leq \|\phi\|_A \|\psi\|_A$, so taking the supremum over all $x \in A$, we obtain the given inequality. In general, the equality may be strict. For instance, if $\phi(x) = x$, $\psi(x) = 1 - x$, and $A = [0, 1]$, then $\phi(x)\psi(x) = x(1 - x)$, so

$$
\|\phi \psi\|_A = \frac{1}{4} < 1 = 1 \cdot 1 = \|\phi\|_A \|\psi\|_A.
$$