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Explain your work
1. (25 points) Compute the following limits:

(a) \( \lim_{x \to 0} \frac{\arctan(x^2)}{x^2} \).
(b) \( \lim_{x \to 0} \frac{\sin(x^3)}{x \arctan(x^2)} \).

Solution: (a) This is an indeterminate form of type \( \frac{0}{0} \). By L’Hospital’s rule,

\[
\lim_{x \to 0} \frac{\arctan(x^2)}{x^2} = \lim_{x \to 0} \frac{2x}{1+x^4}, \quad \text{if this limit exists}
\]
\[
= \lim_{x \to 0} \frac{1}{1 + x^4}
\]
\[
= 1.
\]

(b) We know that \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \).

This implies

\[
\lim_{x \to 0} \frac{\sin(x^3)}{x^3} = \lim_{t \to 0} \frac{\sin t}{t} = 1.
\]

Therefore, by (a),

\[
\lim_{x \to 0} \frac{\sin(x^3)}{x \arctan(x^2)} = \lim_{x \to 0} \frac{\sin(x^3)}{x^3} \left( \frac{\arctan(x^2)}{x^2} \right)^{-1}
\]
\[
= 1 \cdot 1^{-1}
\]
\[
= 1.
\]
2. **(25 points)** Suppose $f: [0, \infty) \to \mathbb{R}$ is differentiable, $f(0) = 0$, and there exists a constant $M > 0$ such that for all $x > 0$,

$$f'(x) \leq M.$$ 

Show that $f(x) \leq Mx$, for all $x \geq 0$.

**Solution:** Let $x > 0$. By the Mean Value Theorem there exists $\xi \in (0, x)$ such that

$$f(x) - f(0) = f'(\xi)(x - 0).$$

Using $f(0) = 0$, we obtain

$$f(x) = f'(\xi)x.$$ 

Since $f'(\xi) \leq M$ and $x > 0$, it follows that $f(x) \leq Mx$. The inequality is clearly true for $x = 0$. $\blacksquare$
3. (25 points) Show that for all $x \neq 0$:

$$e^{-x} > 1 - x + \frac{x^2}{2} - \frac{x^3}{6}.$$  

Solution: Let $f(x) = e^{-x}$. Since the exponential function is infinitely differentiable, so is $f$. By the Chain Rule we have $f^{(n)}(x) = (-1)^n e^{-x}$, hence

$$f^{(n)}(0) = (-1)^n.$$  

Thus we can apply the third-order Taylor’s theorem at 0 to $f$. For $x \neq 0$ this yields

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{e^{-\xi} x^4}{4!},$$

for some $\xi$ between 0 and $x$. Since

$$\frac{e^{-\xi}}{4!} x^4 > 0,$$

it follows that

$$e^{-x} > 1 - x + \frac{x^2}{2} - \frac{x^3}{6},$$

as claimed. ■
4. (25 points) Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable, $f(0) = 0$ and for all $n = 1, 2, \ldots,$

$$f\left(\frac{1}{n}\right) = 0.$$ 

(a) Show that $f'(0) = 0$.

(b) Show that there exists a positive decreasing sequence $(x_n)$ converging to zero such that $f'(x_n) = 0$. [Hint: Look at $f$ on $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ for each $n$.]

(c) Show that $f''(0) = 0$.

Solution: (a) Since $f$ is differentiable at 0, we have

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{n \to \infty} \frac{f\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n \to \infty} 0 = 0.$$  

(*) follows from the sequential characterization of limits.

(b) Let us apply Rolle’s theorem to $f$ on $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ for each $n$. This is legal, since $f$ is differentiable on $\mathbb{R}$ and equals zero at $1/(n+1)$ and $1/n$. For each $n$ we obtain a point $x_n \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$ such that $f'(x_n) = 0$. Since

$$x_{n+1} < \frac{1}{n+1} < x_n,$$

the sequence is decreasing and converges to zero (by the Squeeze Theorem).

(c) Since $f'$ is differentiable and $f'(0) = 0$, we have:

$$f''(0) = \lim_{x \to 0} \frac{f'(x) - f'(0)}{x} = \lim_{n \to \infty} \frac{f'(x_n)}{x_n} = \lim_{n \to \infty} 0 = 0.$$  

(○) again follows from the sequential characterization of limits. ■