SAN JOSE STATE UNIVERSITY

Math 131B, Spring 2006

Sample Final Exam

MAY 2006

Name: XYZ

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EXPLAIN YOUR WORK
1. **(20 points)** Suppose that $f$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$. If $f(0) = 0$ and $0 \leq f'(x) \leq 2f(x)$ for $0 < x < 1$, prove that $f(x) = 0$ for all $0 \leq x \leq 1$.

**Solution:** Let $g(x) = e^{-2x}f(x)$. Then $g$ is continuous on $[0, 1]$, differentiable on $(0, 1)$, and

$$g'(x) = e^{-2x}[f'(x) - 2f(x)] \leq 0.$$ 

It follows that $g(x) \leq g(0) = 0$. Since $g(x)$ is also non-negative, we obtain that $g(x) = 0$, for all $x \in [0, 1]$. Therefore, $f = 0$. 
2. (20 points) Suppose that \((a_n)\) is a non-negative decreasing sequence such that
\[
\sum_{n=1}^{\infty} a_n < \infty.
\]
Show that \(na_n \to 0\), as \(n \to \infty\).

**Solution:** Denote \(c_n = na_n\). Let \(\varepsilon > 0\) be arbitrary. By the Cauchy convergence criterion there exists \(N > 0\) such that for all \(m > n > N\),
\[
\sum_{k=n}^{m} a_k < 2\varepsilon.
\]
In particular,
\[
\sum_{k=n+1}^{2n} a_k < 2\varepsilon.
\]
Since \((a_n)\) is decreasing, it follows that
\[
\sum_{k=n+1}^{2n} a_k \geq na_{2n}.
\]
Therefore, \(c_{2n} < \varepsilon\) for \(n > N\), so \(c_{2n} \to 0\), as \(n \to \infty\). Since
\[
0 \leq c_{2n+1} = (2n + 1)a_{2n+1} \leq (2n + 1)a_{2n} = c_{2n} + a_{2n},
\]
and \(a_k \to 0\), by the squeeze theorem we obtain \(c_{2n+1} \to 0\) as well. Thus \(c_n \to 0\), as \(n \to \infty\).
3. (20 points) Let $f$ be a differentiable function on $[0, 1]$ and assume

$$|f'(x)| \leq M < \infty,$$

for all $x \in (0, 1)$. Let $n$ be a positive integer. Prove that

$$\left| \sum_{i=0}^{n-1} \frac{f\left(\frac{i}{n}\right)}{n} - \int_0^1 f(x) \, dx \right| \leq \frac{M}{2n}.$$

**Solution:** The key idea is to view the above sum as a Riemann sum of $f$ relative to the partition of $[0, 1]$ by the points $i/n$, $0 \leq i \leq n-1$.

By the Mean Value Theorem, $|f'| \leq M$, implies $|f(x) - f(y)| \leq M|x - y|$, for all $x, y \in \mathbb{R}$. Therefore, if $i/n \leq x \leq (i + 1)/n$, then

$$|f(x) - f(i/n)| \leq M \left( x - \frac{i}{n} \right).$$

It follows:

$$\left| \sum_{i=0}^{n-1} \frac{f\left(\frac{i}{n}\right)}{n} - \int_0^1 f(x) \, dx \right| = \left| \sum_{i=0}^{n-1} \frac{f\left(\frac{i}{n}\right)}{n} - \sum_{i=0}^{n-1} \int_{i/n}^{(i+1)/n} f(x) \, dx \right|$$

$$= \left| \sum_{i=0}^{n-1} \int_{i/n}^{(i+1)/n} \left[ f(i/n) - f(x) \right] \, dx \right|$$

$$\leq \sum_{i=0}^{n-1} \int_{i/n}^{(i+1)/n} |f(x) - f(i/n)| \, dx$$

$$\leq \sum_{i=0}^{n-1} \int_{i/n}^{(i+1)/n} M \left( x - \frac{i}{n} \right) \, dx$$

$$= \sum_{i=0}^{n-1} M \left( x - \frac{i}{n} \right)^2 \bigg|_{i/n}^{(i+1)/n}$$

$$= \sum_{i=0}^{n-1} M \left( \frac{1}{2} \left( x - \frac{i}{n} \right)^2 \right)_{i/n}^{(i+1)/n}$$

$$= M \frac{1}{2n^2}$$

$$= \frac{M}{2n}.$$
4. (20 points) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$|f(x)| \leq \frac{C}{1+x^2},$$

for all $x \in \mathbb{R}$, where $C > 0$ is some constant. Define a function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+n).$$

(a) Show that $F$ is continuous and periodic with period 1.

(b) If $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous function with period 1, show that

$$\int_{0}^{1} F(x) \phi(x) \, dx = \int_{-\infty}^{\infty} f(x) \phi(x) \, dx.$$

**Solution:** (a) Since

$$F(x+1) = \sum_{n=-\infty}^{\infty} f(x+1+n) = \sum_{k=-\infty}^{\infty} f(x+k) = F(x),$$

$F$ is periodic with period 1. Therefore, it is enough to show that $F$ is continuous on $[0,1]$. So suppose $x \in [0,1]$. Then for $n > 0$,

$$|f(x+n)| \leq \frac{C}{1+(x+n)^2} \leq \frac{C}{n^2} = M_n.$$

Since the series $\sum_{n=1}^{\infty} 1/n^2$ converges, so does the series $\sum M_n$, where $n$ goes from $-\infty$ to $\infty$ but $n \neq 0$. Therefore, by the Weierstrass $M$-test, the series defining $F$ converges uniformly on $[0,1]$. Since $f_n(x) = f(x+n)$ is continuous on $\mathbb{R}$, $F$ is continuous on $[0,1]$, hence on $\mathbb{R}$, by periodicity.

(b) We have:

$$\int_{-\infty}^{\infty} f(x) \phi(x) \, dx = \sum_{n=-\infty}^{\infty} \int_{n}^{n+1} f(x) \phi(x) \, dx$$

$$= \sum_{n=-\infty}^{\infty} \int_{0}^{1} f(x+n) \phi(x+n) \, dx$$

$$= \sum_{n=-\infty}^{\infty} \int_{0}^{1} f(x+n) \phi(x) \, dx$$

$$= \sum_{n=-\infty}^{\infty} \int_{0}^{1} f(x+n) \phi(x) \, dx$$

$$= \int_{0}^{1} F(x) \phi(x) \, dx.$$
5. **(20 points)** Let \( f_n(x) = \sin nx, \) where \( 0 \leq x \leq 2\pi \) and \( n = 1, 2, 3, \ldots \). Show that \((f_n)\) does not have a subsequence \((f_{n_k})\) which converges for every \( x \in [0, 1] \) by following these steps:

(1) Suppose that such a subsequence \((f_{n_k})\) exists. Show that \((f_{n_k}(x) - f_{n_{k+1}}(x))^2 \to 0\), as \( k \to \infty \).

(2) Justify the statement 

\[
\lim_{k \to \infty} \int_0^{2\pi} (f_{n_k}(x) - f_{n_{k+1}}(x))^2 \, dx = 0.
\]

(3) Show that 

\[
\int_0^{2\pi} (f_{n_k}(x) - f_{n_{k+1}}(x))^2 \, dx = 2\pi.
\]

**Solution:** (1) Assume \((f_{n_k}(x))\) converges for all \( x \in [0, 2\pi] \) to some \( f(x) \). Then \( f_{n_{k+1}}(x) \) clearly has the same limit. Therefore, \( f_{n_k}(x) - f_{n_{k+1}}(x) \to 0 \), hence \((f_{n_k}(x) - f_{n_{k+1}}(x))^2 \to 0\), as \( k \to \infty \).

(2) Since \(|\sin x| \leq 1\), it follows that \( 0 \leq (f_{n_k}(x) - f_{n_{k+1}}(x))^2 \leq 4 \), so we can use Lebesgue’s Dominated Convergence Theorem to switch the limit and the Lebesgue integral, which yields the desired conclusion.

(3) For any integers \( m, n \), we have \(\int_0^{2\pi} \cos nx \, dx = 0\), so

\[
\int_0^{2\pi} \sin^2 mx \, dx = \int_0^{2\pi} \frac{1 - \cos 2mx}{2} \, dx
\]

\[
= \int_0^{2\pi} \frac{1}{2} \, dx
\]

\[
= \pi,
\]

and

\[
\int_0^{2\pi} \sin mx \sin nx \, dx = \int_0^{2\pi} \frac{1}{2} [\cos(m + n)x - \cos(m - n)x] \, dx
\]

\[
= 0.
\]

Therefore,

\[
\int_0^{2\pi} (f_{n_k}(x) - f_{n_{k+1}}(x))^2 \, dx = \int_0^{2\pi} (\sin^2 n_k x - 2(\sin n_k x)(\sin n_{k+1} x) + \sin^2 n_{k+1} x) \, dx
\]

\[
= 2\pi.
\]

Since (3) contradicts (2), there is no subsequence of \((f_n)\) that converges for all \( x \in [0, 2\pi] \).
6. (20 points) Suppose \( f : I \rightarrow [0, \infty) \) is Lebesgue measurable and \( \int_I f < \infty \), where \( I = [0, 1] \). For any \( a > 0 \), let
\[
E_a = \{ x \in I : f(x) > a \}.
\]
Prove that
\[
m(E_a) \leq \frac{1}{a} \int_I f.
\]

Solution: Since \( f \geq 0 \), \( E_a \subset I \), and \( f > a \) on \( E_a \), we obtain:
\[
\int_I f \geq \int_{E_a} f \\
\geq \int_{E_a} a \\
= a \cdot m(E_a).
\]
Dividing by \( a \) we obtain the desired inequality.