1. (hard) Describe the set $S$ of limit points of all convergent subsequences of the sequence $(\sin n)$. (Observe that $S \subseteq [0, 1]$ and $S$ is non-empty, by the Bolzano-Weierstrass theorem.)

2. If
   
   $C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$

   where $C_0, C_1, \ldots, C_n$ are real constants, prove that the equation
   
   $C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n = 0$

   has at least one real root between 0 and 1.

3. We call $p$ a **fixed point** of $f : \mathbb{R} \to \mathbb{R}$ if $f(p) = p$.

   (a) If $f$ is differentiable and $f'(t) \neq 1$ for every real $t$, prove that $f$ has at most one fixed point.

   (b) Suppose there is a constant $\lambda \in (0, 1)$ such that $|f'(t)| \leq \lambda$, for all $t \in \mathbb{R}$. Let $x_0$

   be arbitrary and define a sequence $(x_n)$ by
   
   $x_{n+1} = f(x_n)$.

   Show that $(x_n)$ converges and that its limit is the unique fixed point of $f$.

4. Find an example of a function $f$ such that the limit
   
   $\lim_{h \to 0} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2}$

   exists but $f''(x)$ doesn’t. (Recall it was/will be proved in class that if $f$ is twice

   differentiable, then $f''(x)$ equals this limit.)

5. Suppose that $f : [0, 1] \to \mathbb{R}$ has a continuous derivative on $[0, 1]$ and $f(0) = 0$. Show

   that
   
   $\max_{0 \leq x \leq 1} |f(x)| \leq \left( \int_0^1 |f'(t)|^2 \, dt \right)^{1/2}.$

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