Sec. 5.2, ex. 7: Let \( f : [0, 1] \to \mathbb{R} \) be defined by
\[
f(x) = \begin{cases} 
1, & \text{if } x \in [0, 1] \cap \mathbb{Q} \\
-1, & \text{if } x \in [0, 1] \setminus \mathbb{Q}.
\end{cases}
\]
Then \(|f| = 1\) is clearly continuous. Let us show that \( f \) is discontinuous at every point of \([0, 1]\).
If \( x \in [0, 1] \) is rational, let \((x_n)\) be a sequence of irrationals in \([0, 1]\) converging to \(x\). Such a sequence exists because the set of irrational numbers is dense in \(\mathbb{R}\). Then
\[
\lim_{n \to \infty} f(x_n) = -1 \neq 1 = f(x).
\]
By the characterization of continuity via sequences, it follows that \( f \) is discontinuous at \(x\).
If \( y \) is irrational, let \((y_n)\) be a sequence of rational numbers converging to \(y\). Such a sequence exists because the set of rational numbers is dense in \(\mathbb{R}\). Then
\[
\lim_{n \to \infty} f(y_n) = 1 \neq -1 = f(y).
\]
Therefore, \( f \) is discontinuous at \(y\), hence at every point of \([0, 1]\). \(\square\)

5.3, ex. 6: Let
\[
g(x) = f(x) - f \left( x + \frac{1}{2} \right).
\]
It suffices to show that \( g(c) = 0 \), for some \( 0 \leq c \leq 1/2 \).
Since \( f \) is continuous, so is \( x \mapsto f \left( x + \frac{1}{2} \right) \), as the composition of two continuous functions.
It follows that \( g \) is continuous on its domain \([0, \frac{1}{2}]\). Furthermore, \( g(0) = f(0) - f(1/2) \) and
\[
g \left( \frac{1}{2} \right) = f \left( \frac{1}{2} \right) - f(1) = f \left( \frac{1}{2} \right) - f(0).
\]
If \( g(0) = 0 \), then \( f(0) = f(1/2) \), so we can take \( c = 0 \). Otherwise, \( g(0) \neq 0 \), so \( g(0) \) and \( g(1/2) \) are of opposite sign. By the Intermediate Value Theorem (IVT), it follows that there exists \( c \in (0, 1/2) \) such that \( g(c) = 0 \). \(\square\)

5.3, ex. 17: Suppose \( f \) is not constant. Then it takes at least two distinct values, say \( u < v \).
Since \( f \) is continuous, by the Intermediate Value Theorem, every number in \((u, v)\) is a value of \( f \). But every open interval \((u, v)\) contains both rational and irrational numbers, contradicting the assumption that \( f \) takes only rational or only irrational values. \(\square\)

Sec. 6.1, ex. 4: By definition of \( f \), we have
\[
0 \leq |f(x)| \leq x^2,
\]
for all \( x \in \mathbb{R} \). Therefore,
\[
0 \leq \left| \frac{f(x)}{x} \right| \leq |x|,
\]
for all \( x \neq 0 \). By the Squeeze Theorem, \( |f(x)/x| \to 0 \), as \( x \to 0 \) and therefore, \( f(x)/x = |f(x) - f(0)|/(x - 0) \to 0 \). Thus \( f \) is differentiable at \( 0 \) and \( f'(0) = 0 \). \(\square\)
**Remark:** It is sufficient to show that \( f(x)/x \to 0 \), as \( Q \in x \to 0 \) and \( Q \notin x \to 0 \) separately, but you need to prove the sufficiency.

**Sec. 6.1, ex. 13:** Suppose first that \( f \) is differentiable at \( c \). By definition, the limit
\[
\lim_{h \to 0} \frac{f(c + h) - f(c)}{h} \tag{1}
\]
equals and equals \( f'(c) \). By the sequential characterization of limits, \( f'(c) \) equals the limit in (1) along any sequence \( (h_n) \) such that \( h_n \to 0 \), as \( n \to \infty \). Taking \( h_n = 1/n \) gives the desired conclusion.

To see that the existence of the limit of \( n[f(c + 1/n) - f(c)] \) does not guarantee the existence of \( f'(c) \), take
\[
f(x) = \begin{cases} 
  x \sin \frac{2\pi}{x}, & \text{if } x \neq 0 \\
  0, & \text{if } x = 0.
\end{cases}
\]
Then the limit of
\[
n[f(1/n) - f(0)] = \sin 2n\pi = 0
\]
is 0, yet \( f \) is not differentiable at \( c = 0 \), since
\[
\lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{2\pi}{x}
\]
does not exist. \( \square \)

**Sec. 6.1, ex. 16:** Denote by \( f \) the restriction of tangent to the interval \((-\pi/2, \pi/2)\). Observe that \( f(-\pi/2, \pi/2) = \mathbb{R} \), since the limits of \( f \) at the endpoints are \(-\infty\) and \(+\infty\). Furthermore, \( f \) is differentiable and
\[
f'(x) = \frac{1}{\cos^2 x} > 0.
\]
Therefore, \( f : (-\pi/2, \pi/2) \to \mathbb{R} \) has a differentiable inverse, \( f^{-1} = \arctan : \mathbb{R} \to (-\pi/2, \pi/2) \), whose derivative is
\[
(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{\frac{1}{\cos^2 x}} = \cos^2 x,
\]
where \( x = f^{-1}(y) \), i.e., \( y = \tan x \). Using \( \cos^2 x + \sin^2 x = 1 \), we obtain
\[
\cos^2 x = \frac{\cos^2 x}{1} = \frac{\cos^2 x}{\cos^2 x + \sin^2 x} = \frac{1}{1 + \tan^2 x} = \frac{1}{1 + y^2}. \quad \square
\]

**Sec. 6.2, ex. 4:** Taking the derivative of
\[
f(x) = \sum_{i=1}^{n} (a_i - x)^2 = \sum_{i=1}^{n} (x - a_i)^2,
\]
we obtain, for any \( x \in \mathbb{R} \),
\[
f'(x) = \sum_{i=1}^{n} 2(x - a_i).
\]
Solving \( f'(x) = 0 \), we obtain the only critical point of \( f \)
\[
c = \frac{1}{n} \sum_{i=1}^{n} a_i.
\]
Since \( f'(x) = 2n(x - c) \), observe that
\[
f'(x) \begin{cases} < 0 & \text{if } x < c \\ > 0 & \text{if } x > c. \end{cases} \tag{2}
\]
By the First Derivative Test, \( f \) has a minimum at 0. The minimum is absolute (or global) since by (2) \( f \) is strictly decreasing on \( (-\infty, c] \) and strictly increasing on \([c, \infty)\). Since \( f \) has no other critical points, the Interior Extremum Theorem implies that \( c \) is the unique extremum point of \( f \).

\[ \square \]

**Sec. 6.2, ex. 16:** (a) Let \( h > 0 \) be fixed. By the Mean Value Theorem, there exists a point \( c_\varepsilon \in (x, x + h) \) such that \( f(x + h) - f(x) = f'(c_\varepsilon)h \). Observe that \( c_\varepsilon \to \infty \), as \( x \to \infty \). Therefore,

\[
\lim_{x \to \infty} \frac{f(x + h) - f(x)}{h} = \lim_{x \to \infty} f'(c_\varepsilon) = b.
\]

(b) Suppose that \( f(x) \to a \), as \( x \to \infty \). Then using (a), we have:

\[
b = \lim_{x \to \infty} \frac{f(x + h) - f(x)}{h} = \lim_{x \to \infty} \frac{f(x + h) - f(x)}{h} = \frac{a - a}{h} = 0.
\]

(c) First assume \( b > 0 \). Let \( \varepsilon > 0 \) be arbitrary. Without loss of generality, we can assume that \( \varepsilon < 1 \) and \( \varepsilon < 2b \) (since, after all, we only care for very small values of \( \varepsilon \)). Since \( f'(x) \to b \), as \( x \to \infty \), there exists \( a > 0 \) such that \( |f'(x) - b| < \varepsilon/2 \), for all \( x \geq a \). Put \( K = \max \left( \frac{a}{\varepsilon}, \frac{2f(a)}{\varepsilon} \right) \).

By the Mean Value Theorem, for every \( x > K(> a) \), there exists a number \( \xi \in (a, x) \) such that \( f(x) - f(a) = f'(\xi)(x - a) \). Therefore,

\[
f(x) = \frac{f(x) - f(a)}{x} + f(a)
= f'(\xi)\left(1 - \frac{a}{x}\right) + f(a).
\]

Observe that \( b - \varepsilon/2 < f'(\xi) < b + \varepsilon/2 \). \( 1 - \varepsilon < 1 - \frac{a}{x} < 1 \), and \( -\varepsilon/2 < f(a)/x < \varepsilon/2 \). Using these estimates, we obtain two inequalities:

\[
\frac{f(x)}{x} < \left(b + \frac{\varepsilon}{2}\right) \cdot 1 + \frac{\varepsilon}{2} = b + \varepsilon,
\]

and

\[
\frac{f(x)}{x} > \left(b - \frac{\varepsilon}{2}\right) (1 - \varepsilon) - \frac{\varepsilon}{2} = b - \varepsilon (1 + b) + \frac{\varepsilon^2}{2} > b - \varepsilon (1 + b).
\]

In other words, for all \( x > K \), we have

\[
b - \varepsilon (1 + b) < \frac{f(x)}{x} < b + \varepsilon.
\]

Since \( \varepsilon \) can be arbitrarily small, it follows that \( f(x)/x \to b \), as \( x \to \infty \). If \( b < 0 \) the argument is symmetric, and for \( b = 0 \) it is similar.

\[ \square \]

**Remark:** You may complain that we didn’t show that \( f(x)/x \) was greater than \( b - \varepsilon \), only \( b - \varepsilon (1 + b) \). However, it is not necessary to get exactly \( \varepsilon \) on both sides. Here’s a useful lemma:

**Lemma.** Let \( f : I \to \mathbb{R} \) and \( a \in I \). Suppose that there exist functions \( u, v \) continuous at 0, with \( u(0) = v(0) = 0 \), such that for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that

\[
L - u(\varepsilon) < f(x) < L + v(\varepsilon),
\]

for all \( x \in I \), \( 0 < |x - a| < \delta \). Then \( f(x) \to L \), as \( x \to a \).
If \( u(x) = v(x) = x \), we get our usual definition of the limit at \( f \) at \( a \). Proving this lemma is a good exercise in \( \varepsilon-\delta \) analysis.

**Sec. 6.2, ex. 20:** (a) Applying the Mean Value Theorem to \( f \) on \([0, 1]\), we obtain a point \( c_1 \in (0, 1) \) such that \( f(1) - f(0) = f'(c_1)(1 - 0) \), i.e., \( f'(c_1) = 1 \).

(b) Applying Rolle’s Theorem to \( f \) on \([1, 2]\) yields a point \( c_2 \in (1, 2) \) such that \( f'(c_2) = 0 \).

(c) Since \( f'(c_1) = 1 \), \( f'(c_2) = 0 \), and \( 0 < \frac{1}{3} < 1 \), by Darboux’s theorem there exists a point \( c \) between \( c_1 \) and \( c_2 \) such \( f'(c) = 1/3 \). \( \square \)