Sec. 6.3, ex. 5: Note that
\[
0 \leq \left| \frac{f(x)}{g(x)} \right| = \left| \frac{x^2 \sin \frac{1}{x}}{\sin x} \right| \leq |x| \left| \frac{x}{\sin x} \right|.
\]

Using the fact that \( x/\sin x = (\sin x/x)^{-1} \to 1 \), as \( x \to 0 \), we obtain that the right-hand side goes to 0, as \( x \to 0 \). The Squeeze Theorem yields \( f(x)/g(x) \to 0 \), as \( x \to 0 \).

On the other hand,
\[
\frac{f'(x)}{g'(x)} = \frac{2x \sin \frac{1}{x} + \cos \frac{1}{x}}{\cos x} = \frac{2x \sin \frac{1}{x}}{\cos x} + \frac{\cos \frac{1}{x}}{\cos x} = A + B.
\]

Again by the Squeeze Theorem, \( A \to 0 \), as \( x \to 0 \). However, \( \cos \frac{1}{x} \) hence \( B \) does not have a limit, as \( x \to 0 \), for usual reasons. Namely, let \( x_n = 1/2n\pi \) and \( y_n = (\pi/2 + 2n\pi)^{-1} \). Both sequences converge to zero, as \( n \to \infty \), yet
\[
\lim_{n \to \infty} \cos \frac{1}{x_n} = 1 \neq 0 = \lim_{n \to \infty} \cos \frac{1}{y_n}. \]

Sec. 6.3, ex. 10: (a) Using the fact \( \log x/x \to 0 \), as \( x \to \infty \), we obtain
\[
\lim_{x \to \infty} x^{1/x} = \lim_{x \to \infty} \exp \left( \frac{\log x}{x} \right)
= \exp \left( \lim_{x \to \infty} \frac{\log x}{x} \right), \quad \text{by continuity of exp}
= \exp(0)
= 1.
\]

(b) Since \( \sin x/x \to 1 \), as \( x \to 0 \), it follows that
\[
\lim_{x \to 0} (\sin x)^x = \lim_{x \to 0} \left\{ \left( \frac{\sin x}{x} \right)^x \right\}
= \lim_{x \to 0} \left( \frac{\sin x}{x} \right)^x \lim_{x \to 0} x^x
= 1^0 \cdot 1
= 1.
\]

(c) Similarly,
\[
\lim_{x \to 0} x^{\sin x} = \lim_{x \to 0} (x^x)^{\frac{\sin x}{x}}
= 1^1
= 1.
\]

(d) Note that
\[
F(x) = \sec x - \tan x = \frac{1 - \sin x}{\cos x} = \frac{f(x)}{g(x)}.
\]
As \( x \to \frac{\pi}{2}^− \), this is an indeterminate form of type \( \frac{0}{0} \). Since
\[
\lim_{x \to \pi/2^-} \frac{f'(x)}{g'(x)} = \lim_{x \to \pi/2^-} \frac{-\cos x}{-\sin x} = 0,
\]
and \( \sin x \neq 0 \), for \( x \in (0, \pi) \) (which is a neighborhood of \( \pi/2 \)), L'Hospital's rule implies that \( F(x) \to 0 \), as \( x \to \pi/2^- \). \( \square \)

**Sec. 6.3, ex. 11:** Let us apply L'Hospital's rule to \( e^x f(x)/e^x(= f(x)) \). Since \( e^x \to \infty \), as \( x \to \infty \), \((e^x f(x))' = e^x f(x) + e^x f'(x)\), and
\[
\lim_{x \to \infty} \frac{e^x f(x) + e^x f'(x)}{e^x} = \lim_{x \to \infty} [f(x) + f'(x)] = L,
\]
we conclude that L'Hospital's rule does apply and \( f(x) \to L \), as \( x \to \infty \). Therefore,
\[
f'(x) = [(f(x) + f'(x)) − f(x)] \to L − L = 0, \quad \text{as } x \to \infty. \quad \square
\]

**Sec. 6.4, ex. 3:** We will use induction on \( n \geq 0 \) to prove that
\[
(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}. \tag{1}
\]

**Induction base** For \( n = 0 \) this is clear, since \( \binom{0}{0} = 1 \) and the right-hand side is just \( fg \).

**Induction step** Assume the identity holds for some \( n \geq 0 \) (induction hypothesis). Suppose \( f, g \) are \( (n + 1) \)-times differentiable. Then:
\[
(fg)^{(n+1)} = \left\{ (fg)^{(n)} \right\}'
\]
\[
= \left\{ \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)} \right\}', \quad \text{by induction hypothesis}
\]
\[
= \sum_{k=0}^{n} \left\{ \binom{n}{k} f^{(n+1-k)} g^{(k)} + \binom{n}{k} f^{(n-k)} g^{(k+1)} \right\}, \quad \text{by the product rule}
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} f^{(n+1-k)} g^{(k)} + \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k+1)}
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} f^{(n+1-k)} g^{(k)} + \sum_{\ell=1}^{n+1} \binom{n}{\ell-1} f^{(n+1-\ell)} g^{(\ell)}, \quad \text{where } \ell = k + 1
\]
\[
= f^{(n+1)} g + \sum_{k=1}^{n} \left[ \binom{n}{k} + \binom{n}{k-1} \right] f^{(n+1-k)} g^{(k)} + f g^{(n+1)} \tag{*}
\]
\[
= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(n+1-k)} g^{(k)}, \quad \text{by the Lemma.}
\]
To get \((*)\), set \( k = \ell \) in the second sum, separate the \( k = 0 \) term from the first sum and \( k = n+1 \) term from the second one, and combine the remaining sums.

**Lemma.**
\[
\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}.
\]
Proof. We have:

\[
\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n+1-k)!} = \frac{(n+1-k)n! + kn!}{k!(n+1-k)!} = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}.
\]

By the principle of mathematical induction, the proof of (1) is complete. \□

Sec. 6.4, ex. 10: First of all, on \( \mathbb{R} \setminus \{0\} \), \( h \) is composed of two infinitely differentiable functions, hence it is infinitely differentiable itself. At \( x = 0 \), however, we need to use the definition of the derivative to compute \( h^{(n)}(0) \).

We claim that for each \( n = 1, 2, 3, \ldots \), there exists a polynomial \( Q_n(t) \) such that for all \( x \neq 0 \),

\[
h^{(n)}(x) = e^{-1/x^2} Q_n \left( \frac{1}{x} \right).
\] (2)

The proof goes by induction. For \( n = 1 \), we have

\[
h'(x) = e^{-1/x^2} \frac{2}{x^3},
\]

so (2) is true, with \( Q_1(t) = 2t^3 \). Assume (2) is true for some \( n \geq 1 \). Then

\[
h^{(n+1)}(x) = \left( h^{(n)}(x) \right)' = \left( e^{-1/x^2} Q_n \left( \frac{1}{x} \right) \right)' = e^{-1/x^2} \left\{ \frac{2}{x^3} Q_n \left( \frac{1}{x} \right) - \frac{1}{x^2} Q'_n \left( \frac{1}{x} \right) \right\}.
\]

Therefore, if we take \( Q_{n+1}(t) = 2t^3 Q_n(t) - t^2 Q'_n(t) \), we obtain \( h^{(n+1)}(x) = e^{-1/x^2} Q_{n+1} \left( \frac{1}{x} \right) \), and the claim follows.

Next, we show that \( h(x)/x^n \to 0 \), as \( x \to 0 \) (\( n \in \mathbb{N} \)). That is, \( h \) is infinitely flat at 0 (it’s flatter at 0 than any monomial \( x^n \)). We will use the fact that \( t^k/e^{t^2} \to 0 \), as \( t \to \infty \), for all \( k \geq 0 \). This follows from L’Hospital’s rule and was proved in class. Thus

\[
\frac{h(x)}{x^n} = \frac{e^{-1/x^2}}{x^n} = \frac{e^{-t^2}}{t^{-n}}, \quad \text{where } x = 1/t
\]

as \( x \to 0 \) (hence \( t \to \infty \)).

Let us now prove that \( h^{(n)}(0) = 0 \), for all \( n \geq 0 \). We again use induction. For \( n = 0 \), this is true by definition. Assume \( h^{(n)}(0) = 0 \), for some \( n \geq 0 \). Then, using (2) and assuming
\( Q_n(t) = \sum_k a_k t^k \), we obtain

\[
\frac{h^{(n)}(x) - h^{(n)}(0)}{x} = e^{-1/x^2} Q_n \left( \frac{1}{x} \right) \\
= \frac{1}{x} \sum_k a_k e^{-1/x^2} \frac{1}{x^k} \\
= \sum_k a_k \frac{e^{-1/x^2}}{x^{k+1}}.
\]

We just saw that all the terms in this finite sum go to zero, as \( x \to 0 \). Therefore, the sum goes to zero as well, and by definition of the derivative, \( h^{(n+1)}(0) = 0 \). By the principle of mathematical induction, \( h^{(n)}(0) = 0 \), for all \( n \geq 0 \).

Let \( P_n \) be the \( n \)th order Taylor polynomial of \( h \) based at 0 and let \( R_n = h - P_n \) be the corresponding remainder. Assume \( R_n(x) \) does converge to 0, for some \( x \neq 0 \). Then

\[
\lim_{n \to \infty} \frac{h(x)}{x} = \lim_{n \to \infty} \frac{P_n(x) + R_n(x)}{x} = \lim_{n \to \infty} P_n(x).
\]

But since all derivatives of \( h \) at zero equal zero, \( P_n = 0 \), for all \( n \). Hence \( h(x) = 0 \), a contradiction. Therefore, \( R_n(x) \) does not converge to zero. \( \Box \)

**Moral.** There exists an infinitely differentiable function that does not equal the sum of its Taylor series. Denote by \( C^\infty \) the class of infinitely differentiable functions and by \( C^\omega \) the class of analytic functions (those that are equal to the sum of their Taylor series; we’ll talk about them more later, when we cover Chapter 9). We know/will soon learn that \( C^\omega \subseteq C^\infty \). This very important example tells us that \( C^\omega \) is properly contained in \( C^\infty \), i.e.,

\[
C^\infty \subset C^\omega \quad \text{but} \quad C^\infty \neq C^\omega.
\]

**Sec. 6.4, ex. 18: Proof using Taylor.** Since \( g^{(n)} \) is continuous and \( g^{(n)}(c) \neq 0 \), there exists a neighborhood \( U \) of \( c \) such that \( g^{(n)}(x) \neq 0 \), for all \( x \in U \). By Taylor’s theorem, for every \( x \in U \), there exist numbers \( \xi \) and \( \eta \) between \( x \) and \( c \) such that

\[
f(x) = P_{n-1}(x) + \frac{f^{(n)}(\xi)}{n!}(x-c)^n
\]

and

\[
g(x) = Q_{n-1}(x) + \frac{g^{(n)}(\eta)}{n!}(x-c)^n,
\]

where \( P_{n-1} \) and \( Q_{n-1} \) are the \((n-1)\)th-order Taylor polynomials of \( f, g \) at \( c \), respectively. However, \( P_{n-1} = 0 \) and \( Q_{n-1} = 0 \). Therefore, for \( x \in U \setminus \{c\} \),

\[
\frac{f(x)}{g(x)} = \frac{f^{(n)}(\xi)}{g^{(n)}(\eta)} \to \frac{f^{(n)}(c)}{g^{(n)}(c)},
\]

as \( x \to c \), since both \( f^{(n)} \) and \( g^{(n)} \) are continuous and \( \xi, \eta \to c \). \( \Box \)

**Proof using L’Hospital.** We will prove the statement by induction on \( n \geq 1 \) (again!). For \( n = 1 \) (induction base), this is just the statement of the “preliminary L’Hospital” (Theorem 6.3.1), as long as we can show that \( g(x) \neq 0 \), for all \( x \) in some neighborhood of \( c \) (and \( x \neq c \)).
Suppose not, i.e., suppose that there is no neighborhood of \( c \) on which \( g \neq 0 \). Then there must exist a sequence \( (x_k) \) converging to \( c \) such that \( g(x_k) = 0 \), for all \( k \). But then

\[
g'(c) = \lim_{k \to \infty} \frac{g(x_k) - g(c)}{x_k - c} = \lim_{k \to \infty} \frac{0 - 0}{x_k - c} = 0,
\]

contrary to our assumption. Therefore, for \( n = 1 \) the statement follows from Theorem 6.3.1.

Now assume the statement of the problem holds for some \( n \geq 1 \) (induction hypothesis). Assume \( f \) and \( g \) are \((n+1)\) times differentiable on \( I \) and \( f^{(k)}(c) = 0 = g^{(k)}(c) \), for \( k = 0, 1, \ldots, n \), but \( g^{(n+1)}(c) \neq 0 \).

Put \( F = f' \) and \( G = g' \). Then \( F^{(k)}(c) = 0 = G^{(k)}(c) \), for \( k = 0, \ldots, n-1 \), but \( G^{(n)}(c) = g^{(n+1)}(c) \neq 0 \). By the induction hypothesis,

\[
\lim_{x \to c} \frac{f'(x)}{g'(x)} = \lim_{x \to c} \frac{F(x)}{G(x)} = \frac{F^{(n)}(c)}{G^{(n)}(c)} = \frac{f^{(n+1)}(c)}{g^{(n+1)}(c)}.
\]

To conclude that the limit as \( x \to c \) of \( f(x)/g(x) \) equals the limit of \( f'(x)/g'(x) \), we will use L’Hospital’s rule. However, we need to make sure that \( g(x) \neq 0 \) in some neighborhood of \( c \) (all the other assumptions are clearly satisfied). Suppose not. Then there exists a sequence \( (x_k) \) converging to \( c \) such that \( g(x_k) = 0 \), for all \( k \). Without loss we can assume that all the points \( x_k \) are different from each other. Applying Rolle’s theorem to \( g \) on the intervals with endpoints \( x_k, x_{k+1} \), we obtain a sequence \( x^{(1)}_k \to 0 \) such that \( g'(x^{(1)}_k) = 0 \). Applying the same procedure to the sequence \( (x^{(1)}_k) \), we obtain a sequence \( x^{(2)}_k \to 0 \) such that \( g''(x^{(2)}_k) = 0 \), etc. Eventually, we get a sequence \( x^{(n)}_k \to 0 \) (as \( k \to \infty \)) such that \( g^{(n)}(x^{(n)}_k) = 0 \). Thus

\[
g^{(n+1)}(0) = \lim_{k \to \infty} \frac{g^{(n)}(x^{(n)}_k) - g^{(n)}(c)}{x^{(n)}_k - c} = 0,
\]

which contradicts the assumption \( g^{(n+1)}(c) \neq 0 \). This completes the proof. \( \square \)

**Remark:** The second proof does not use continuity of \( f^{(n)} \) and \( g^{(n)} \).