Sec. 11.2, ex. 1: Let \( U_n = (1 + \frac{1}{n}, 3) \), \( n \geq 1 \). Then \( \mathcal{U} = \{ U_n \} \) is an open cover of \( (1, 2] \). We claim that \( \mathcal{U} \) has no finite subcover. Suppose that it did and denote it by \( \{ U_{n_1}, \ldots, U_{n_k} \} \). Let \( n = \max\{ n_1, \ldots, n_k \} \). Then

\[ U_{n_1} \cup \ldots \cup U_{n_k} = \left( 1 + \frac{1}{n}, 3 \right) = U_n, \]

which clearly does not cover \( (1, 2] \) – a contradiction. Therefore, \( \mathcal{U} \) has no finite subcover, which means that \( (1, 2] \) is not compact. \( \square \)

Sec. 11.2, ex. 3: Let \( V_n = (\frac{1}{n+1}, 2) \), \( n \geq 1 \). Since \( V_n \) contains \( 1, \frac{1}{2}, \ldots, \frac{1}{n} \), \( \mathcal{V} = \{ V_n \} \) is an open cover of the set \( 1/\mathbb{N} = \{ 1/n : n \in \mathbb{N} \} \). We claim that \( \mathcal{V} \) has no finite subcover. Suppose this were not true. Denote a finite subcover of \( \mathcal{V} \) by \( \{ V_{n_1}, \ldots, V_{n_k} \} \). Let \( n = \max\{ n_1, \ldots, n_k \} \). Then

\[ V_{n_1} \cup \ldots \cup V_{n_k} = \left( \frac{1}{n}, 2 \right) = V_n, \]

which clearly does not cover \( 1/\mathbb{N} \) – a contradiction. Therefore, \( \mathcal{V} \) has no finite subcover, so \( 1/\mathbb{N} \) is not compact. \( \square \)

Sec. 11.2, ex. 8: Let \( \{ K_i : i \in I \} \) be an arbitrary collection of compact subsets of \( \mathbb{R} \) and set

\[ K = \bigcap_{i \in I} K_i. \]

By the Heine-Borel theorem, each \( K_i \) is closed and bounded. The intersection of an arbitrary collection of closed sets is closed, so \( K \) is closed. Since \( K \subseteq K_1 \) and \( K_1 \) is bounded, it follows that \( K \) is bounded. Therefore, \( K \) is both closed and bounded, hence compact, by Heine-Borel. \( \square \)

Sec. 11.2, ex. 10: Since \( K \) is compact, it is bounded, by Heine-Borel. Therefore, \( K \) has an infimum and a supremum, and they are finite, since \( K \neq \emptyset \) (recall that \( \inf \emptyset = +\infty \) and \( \sup \emptyset = -\infty \)). Let \( a = \inf K \) and \( b = \sup K \). By one of the characterizations of the infimum, in addition to \( a \) being a lower bound of \( K \), there exists a sequence \( \langle a_n \rangle \) in \( K \) such that \( a_n \to a \). If this sequence is eventually constant – i.e., if \( a_N = a_{N+1} = a_{N+2} \cdots \), starting from some \( N \) – then \( a = a_N \) is clearly in \( K \). Otherwise, \( a \) is a cluster point of \( K \). Since \( K \) is compact, it is closed, and it therefore contains all of its cluster points. Thus \( a \in K \). The proof that \( b \in K \) is analogous (or one can also use the fact \( \sup K = -\inf(-K) \)). \( \square \)

Sec. 11.3, ex. 5: Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is continuous and let \( \alpha \in \mathbb{R} \) be arbitrary. We have

\[ U_\alpha = \{ x \in \mathbb{R} : f(x) < \alpha \} = f^{-1}(\langle -\infty, \alpha \rangle). \]

Since \( (-\infty, \alpha) \) is open (it’s an open interval), it follows that \( U_\alpha \) is open, as the preimage of an open set by a continuous function. \( \square \)

Sec. 11.3, ex. 10: Let

\[ F = \{ x \in I : f(x) = g(x) \}. \]
Let $x$ be an arbitrary cluster point of $F$. Then there exists a sequence $(x_n)$ in $F$ such that $x_n \to x$, as $n \to \infty$. Since $x_n \in F$, for every $n \in \mathbb{N}$, we have

$$f(x_n) = g(x_n).$$

Letting $n \to \infty$ and using the sequential characterization of continuity of $f$ and $g$, we obtain $f(x_n) \to f(x)$ and $g(x_n) \to g(x)$. This implies that $f(x) = g(x)$, so $x \in F$, by definition of $F$. Since $F$ contains all of its cluster points, it is closed. \qed