Name: Karl Weierstrass

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1. **(25 points)** If \( f : [0, 1] \to \mathbb{R} \) is a continuous function and

\[
\int_0^{t^2} f = \int_{t^2}^1 f,
\]

for all \( t \in [0, 1] \), show that \( f(x) = 0 \), for all \( x \in [0, 1] \).

**Solution:** Since \( f \) is continuous, by the Fundamental Theorem of Calculus both sides of the above identity are differentiable. Using the Chain Rule and FTC, we obtain

\[
\frac{d}{dt} \int_0^{t^2} f = 2tf(t^2)
\]

and

\[
\frac{d}{dt} \int_{t^2}^1 f = -\frac{d}{dt} \int_1^{t^2} f = -2tf(t^2).
\]

Therefore,

\[
2tf(t^2) = -2tf(t^2),
\]

for all \( t \in [0, 1] \). This implies \( f(t^2) = 0 \), for all \( t \in (0, 1] \), that is, \( f = 0 \) on \((0, 1] \). Continuity of \( f \) implies that \( f(0) = 0 \). Therefore, \( f = 0 \).
2. (25 points) Let
\[ f_n(x) = nx(1 - x)^n, \]
for \( n \in \mathbb{N} \) and \( x \in [0, 1] \).

(a) Find the pointwise limit of the sequence \((f_n)\).
(b) Does \((f_n)\) converge uniformly on \([0, 1]\)?

**Solution:**
(a) If \( x = 0 \) or \( x = 1 \), then \( f_n(x) = 0 \to 0 \). Recall that \( nq^n \to 0 \), as \( n \to \infty \), for \( q \in (0, 1) \). Thus if \( 0 < x < 1 \), then \( 0 < 1 - x < 1 \) and
\[ f_n(x) = nx(1 - x)^n \to 0. \]
Therefore, the pointwise limit of \((f_n)\) on \([0, 1]\) is the zero function.

(b) Let us examine the norm \( \|f_n\|_{[0,1]} = \|f_n - 0\|_{[0,1]} \). We have
\[ f'_n(x) = n(1 - x)^{(n-1)}[1 - (n + 1)x], \]
which equals zero iff \( x = 0 \) or \( x = 1/(n + 1) \). Since \( f_n(0) = 0 \) and \( f_n \) is nonnegative, it follows that \( f_n \) attains its maximum at \( 1/(n + 1) \) and that maximum equals
\[ \|f_n\|_{[0,1]} = f_n \left( \frac{1}{n + 1} \right) = \left( \frac{n}{n + 1} \right)^{n+1} = \left( 1 + \frac{-1}{n + 1} \right)^{n+1} \to e^{-1}, \]
as \( n \to \infty \). (Recall that \( (1 + x/n)^n \to e^x \), as \( n \to \infty \).) Therefore, \((f_n)\) does *not* converge uniformly on \([0, 1]\).
3. (25 points) (a) Assume $u_n \leq a_n \leq v_n$, for all $n \in \mathbb{N}$, where

$$u_n = \begin{cases} 1, & \text{if } n \text{ is odd} \\ n, & \text{otherwise} \end{cases}, \quad v_n = \begin{cases} 1, & \text{if } n \text{ is odd} \\ n^2, & \text{otherwise}. \end{cases}$$

Find the radius of convergence of the power series $\sum a_n x^n$.

(b) What is the radius of convergence of $\sum (n + 2)(n + 1)a_{n+2}x^n$?

**Solution:** (a) First recall that $\sqrt[n]{n} \rightarrow 1$, as $n \rightarrow \infty$.

The sequence $(\sqrt[n]{u_n})$ consists of two complementary subsequences: (1) and $(\sqrt[n]{2k})$, which both converge to 1. Therefore,

$$\lim_{n \to \infty} \sqrt[n]{u_n} = 1.$$

Observe that $v_n = u_n^2$. Thus,

$$\lim_{n \to \infty} \sqrt[n]{v_n} = \lim_{n \to \infty} \sqrt[n]{u_n^2} = 1^2 = 1.$$

Since $\sqrt[n]{u_n} \leq \sqrt[n]{a_n} \leq \sqrt[n]{v_n}$, by the Squeeze Theorem we have

$$\lim_{n \to \infty} \sqrt[n]{a_n} = 1.$$

Therefore, the radius of convergence of $\sum a_n x^n$ equals

$$R = \frac{1}{\limsup_{n \to \infty} \sqrt[n]{a_n}} = 1.$$

(b) The series

$$\sum_{n=0}^{\infty} (n + 2)(n + 1)a_{n+2}x^n = 2a_2 + 6a_3x + \cdots$$

is the second derivative of the series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$, so they have the same radius of convergence $R = 1$. 

4. (25 points) Compute
\[
\lim_{n \to \infty} \int_{1/2}^{2} xe^{-nx^3} \, dx.
\]

(Extra credit: 5 points) Compute
\[
\lim_{n \to \infty} \int_{0}^{2} e^{-nx^3} \, dx.
\]

Solution: Let \( f_n(x) = xe^{-nx^3} \). If \( x \in [1/2, 2] \), then
\[
0 \leq f_n(x) \leq 2e^{-\frac{8}{3}}.
\]
Therefore,
\[
\|f_n\|_{[1/2,2]} \leq 2e^{-\frac{8}{3}} \to 0,
\]
as \( n \to \infty \). This shows that \( f_n \to 0 \), uniformly on \([1/2, 2]\). By the theorem on the interchange of the limit and the Riemann integral,
\[
\lim_{n \to \infty} \int_{1/2}^{2} f_n = \int_{1/2}^{2} \lim_{n \to \infty} f_n = \int_{1/2}^{2} 0 = 0.
\]

(Extra credit) Let \( g_n(x) = e^{-nx^3} \). Then:
\[
g(x) := \lim_{n \to \infty} g_n(x) = \begin{cases} 
1, & \text{if } x = 0 \\
0, & \text{if } x \in (0,2]. 
\end{cases}
\]
Therefore, \( (g_n) \) does not converge uniformly on \([0, 2]\), since if it did, the limit would be continuous, as the limit of a sequence of continuous functions. Hence, we cannot apply the theorem we used in the first part of the problem.

However, each \( g_n \) is Riemann integrable (being continuous), as is \( g \) (being discontinuous at a single point and bounded). Furthermore, \( 0 \leq g_n \leq 1 \), so the sequence \( (g_n) \) is uniformly bounded. By the Bounded Convergence Theorem we \emph{can} interchange the limit and the integral:
\[
\lim_{n \to \infty} \int_{0}^{2} g_n = \int_{0}^{2} g = \int_{0}^{2} 0 = 0.
\]