Name: Granwyth Hulatberi

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Explain your work
1. (20 points) Let \( f : \mathbb{R} \to \mathbb{R} \) be a function and let \((a_n)\) and \((b_n)\) be sequences of numbers converging to zero. Assume \(a_n \neq b_n\) for all \(n \in \mathbb{N}\) and define the difference quotients

\[
D_n = \frac{f(b_n) - f(a_n)}{b_n - a_n}.
\]

Prove that \(D_n \to f'(0)\), as \(n \to \infty\), under each of the following conditions (separately):

(a) If \(f\) is continuously differentiable (i.e., \(f'\) exists and is continuous).

(b) If \(f\) is differentiable at zero and \(a_n < 0 < b_n\) for all \(n \in \mathbb{N}\).

(Hint: You may – or may not – want to use the following basic fact: if \(f\) is differentiable at \(a\), then \(f(x) = f(a) + (x-a)f'(a) + \rho(x)\), where \(\rho(x)/(x-a) \to 0\), as \(x \to a\).)

Solution: (a) If \(f\) is \(C^1\), then by the Mean Value Theorem for each \(n\) there exists a number \(c_n\) between \(a_n\) and \(b_n\) such that

\[
f(b_n) - (a_n) = f'(c_n)(b_n - a_n).
\]

Since both \(a_n\) and \(b_n\) converge to zero, so does \(c_n\). Therefore,

\[
D_n = f'(c_n) \to f'(0),
\]

as \(n \to \infty\), by continuity of \(f'\).

(b) By definition of differentiability (and by the Hint), for each \(n \in \mathbb{N}\) there exist \(u_n = \rho(a_n), v_n = \rho(b_n)\) such that

\[
f(a_n) = f(0) + a_nf'(0) + u_n
\]

\[
f(b_n) = f(0) + b_nf'(0) + v_n,
\]

and satisfying \(|u_n/a_n| \to 0\) and \(|v_n/b_n| \to 0\). Therefore,

\[
D_n = f'(0) + \frac{v_n - u_n}{b_n - a_n}.
\]

Since \(a_n < 0 < b_n\), we have \(b_n - a_n > b_n\) and \(b_n - a_n > |a_n|\). This yields

\[
0 \leq \frac{|v_n|}{b_n - a_n} \leq \frac{|v_n|}{b_n} \to 0 \quad \text{and} \quad 0 \leq \frac{|u_n|}{b_n - a_n} \leq \frac{|u_n|}{a_n} \to 0.
\]

By the squeeze theorem,

\[
\frac{v_n - u_n}{b_n - a_n} \to 0,
\]

as \(n \to \infty\), hence \(D_n \to f'(0)\).
2. (20 points) Consider
\[ f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^\alpha x}, \quad x \geq 0. \]

(a) For what values of \( x \) and the parameter \( \alpha \) does the series converge absolutely?
(b) On what intervals does it converge uniformly?
(c) Is \( f \) continuous wherever the series converges?

Solution: (a) First note that the terms of the series are positive, so the series converges iff it converges absolutely.
If \( x > 0 \), then
\[ \frac{1}{1 + n^\alpha x} \leq \frac{1}{n^\alpha}. \]
The series \( \sum n^{-\alpha} \) converges for \( \alpha > 1 \), so the original series converges, by the comparison test. It does not converge for \( x = 0 \) or \( \alpha \leq 0 \), since then \( 1/(1 + n^\alpha x) \) does not converge to zero.
The remaining case to consider is \( 0 < \alpha \leq 1 \) and \( x > 0 \). Then, since \( 1 + n^\alpha x \leq (n+1)^\alpha x \), for large \( n \), we have
\[ \frac{1}{1 + n^\alpha x} \geq \frac{1}{(n+1)^\alpha x}. \]
By comparison test, the series diverges, since \( \sum (n+1)^{-\alpha} \) diverges for \( \alpha \leq 1 \).
Therefore, the series converges iff \( x > 0 \) and \( \alpha > 1 \).

(b) Let \( b > 0 \) be arbitrary but fixed. If \( x \geq b \), then
\[ \frac{1}{1 + n^\alpha x} \leq \frac{1}{bn^\alpha}. \]
Since for \( \alpha > 1 \), the series \( \sum n^{-\alpha} \) converges (absolutely), the given series converges uniformly on every interval \([b, \infty)\), by the Weierstrass M-test.
However, note that the series does not converge uniformly on \((0, \infty)\), since the sequence of functions \( (1 + n^\alpha x)^{-1} \) doesn’t uniformly converge to zero on that set.

(c) For \( \alpha > 1 \), the given series converges uniformly on every interval of the form \([b, \infty)\), for \( b > 0 \). Since each function \( x \mapsto (1 + n^\alpha x)^{-1} \) is continuous on \([b, \infty)\), it follows that \( f \) is continuous on each interval \([b, \infty)\). Therefore, it is continuous on their union, which is \((0, \infty)\). This means that \( f \) is continuous wherever (and whenever) the series converges.
3. **(20 points)** Let $f : [0, 1] \to \mathbb{R}$ be a Riemann integrable function. Show that:

$$\lim_{n \to \infty} \int_0^1 x^n f(x) \, dx = 0.$$ 

**Solution:** Let $g_n(x) = x^n f(x)$. Then

$$g(x) = \lim_{n \to \infty} g_n(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ f(1) & \text{if } x = 1. \end{cases}$$

Since $g$ is bounded and discontinuous at at most one point ($x = 1$), $g$ is Riemann integrable, by the Riemann-Lebesgue criterion. Also, its integral equals zero. By the same criterion, we have:

(a) $g_n$ is Riemann integrable, since $x^n$ is continuous and $f \in \mathcal{R}[0, 1]$,

(b) $f$ is bounded, say, $|f(x)| \leq M$, for all $0 \leq x \leq 1$.

It follows that

$$|g_n(x)| \leq M x^n \leq M,$$

for all $x \in [0, 1]$ and all $n \in \mathbb{N}$, i.e., the sequence $(g_n)$ is uniformly bounded. By the Bounded Convergence Theorem,

$$\lim_{n \to \infty} \int_0^1 g_n(x) \, dx = \int_0^1 g(x) \, dx = 0.$$
4. **(20 + 5 points)** A metric (or distance) on a set $M$ is a function $d : M \times M \to [0, \infty)$ with the following properties: for all $x, y, z \in M$,

- **(D1)** $d(x, y) = 0$ if and only if $x = y$;
- **(D2)** $d(y, x) = d(x, y)$;
- **(D3)** $d(x, z) \leq d(x, y) + d(y, z)$.

A metric space $(M, d)$ is a set $M$ equipped with a metric $d$. A sequence $(x_n)$ converges in $(M, d)$ if there exists $x \in M$ such that $d(x_n, x) \to 0$, as $n \to \infty$. A sequence $(x_n)$ is called Cauchy in $(M, d)$ if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n > N$, $d(x_m, x_n) < \varepsilon$. A metric space $(M, d)$ is called complete if every Cauchy sequence in $(M, d)$ converges to a limit in $M$. See 11.4 in Bartle and Sherbert.

For two functions $f, g : [0, 1] \to \mathbb{R}$, define

$$d_1(f, g) = \int_0^1 |f(t) - g(t)| \, dt \quad \text{and} \quad d_\infty(f, g) = \sup_{0 \leq x \leq 1} |f(t) - g(t)|.$$ 

(a) Show that $d_1$ is a metric on the space $C[0,1]$ of continuous functions defined on $[0,1]$.

(b) Show that $d_1$ is not a metric on the space $\mathcal{R}[0,1]$ of Riemann integrable functions.

(c) Let $f_n(t) = t^n$ ($0 \leq t \leq 1$, $n \in \mathbb{N}$). Show that $(f_n)$ is a Cauchy sequence in $(C[0,1], d_1)$ but that it does not converge to a limit in $C[0,1]$. Therefore, the metric space $(C[0,1], d_1)$ is not complete. (Observe that $(C[0,1], d_\infty)$ is complete by the Cauchy Criterion for Uniform Convergence, 8.1.10).

(d) **(5 extra credit points)** Find a sequence of Riemann integrable functions on $[0,1]$ that is Cauchy with respect to $d_1$ but which does not converge to a Riemann integrable function.

**Solution:** (a) Suppose $d_1(f, g) = 0$ for some $f, g \in C[0,1]$. Then $\int_0^1 |f - g| = 0$. By exercise 10 from 7.2 (see Homework 3 solutions), $f = g$. (D2) is clear. (D3) follows by integrating the triangle inequality for absolute value: $|f - h| \leq |f - g| + |g - h|$ and using the monotonicity of the Riemann integral $u \leq v \Rightarrow \int_a^b u \leq \int_a^b v$.

(b) Let $f$ be the zero function on $[0,1]$ and let $g$ equal zero everywhere but at zero, where we set $g(0) = 1$. By the Riemann-Lebesgue criterion, $f, g \in \mathcal{R}[0,1]$. Since $f$ and $g$ differ at only one point,

$$d_1(f, g) = \int_0^1 |f - g| = \int_0^1 0 = 0,$$

even though $f \neq g$. Therefore, (D1) is violated.
(c) Let \( m > n \) be arbitrary. Then \( t^m \leq t^n \), for \( t \in [0, 1] \), and

\[
d_1(f_m, f_n) = \int_0^1 (t^n - t^m) \, dt = \frac{1}{n+1} - \frac{1}{m+1} \leq \frac{1}{n+1} \to 0,
\]
as \( n \to \infty \). Therefore, \((f_n)\) is a Cauchy sequence in \((C[0, 1], d_1)\). Since

\[
\lim_{n \to \infty} f_n(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & t = 1 \end{cases}
\]
is discontinuous, \((f_n)\) diverges in \(C[0, 1]\).

(d) Since \( \mathbb{Q} \cap [0, 1] \) is countable, it can be written as \( \{r_1, r_2, \ldots\} \). Define \( f_n \) to be the characteristic function of \( \{r_1, \ldots, r_n\} \). Each \( f_n \) is bounded and zero (hence continuous) except at \( n \) points, hence Riemann integrable with integral zero. Therefore,

\[
d_1(f_m, f_n) = \int_0^1 |f_m - f_n| = 0,
\]
for all \( m, n \in \mathbb{N} \), so the sequence \((f_n)\) is Cauchy with respect to \( d_1 \). However, the limit of \( f_n \) is the Dirichlet function (the characteristic function of \( \mathbb{Q} \cap [0, 1] \)), which is not Riemann integrable.
5. (20 points) Suppose that \( f : [a, b] \to \mathbb{R} \) is continuously differentiable, \( f(a) = f(b) = 0 \) and
\[
\int_a^b f(x)^2 \, dx = 1.
\]
Prove that
\[
\int_a^b x f(x) f'(x) \, dx = -\frac{1}{2}.
\]

Proof: This one was easy.

Solution 1: We will use the integration by parts formula
\[
\int_a^b u \, dv = uv\big|_a^b - \int_a^b v \, du,
\]
with \( u = xf(x) \) and \( dv = f'(x) \, dx \). Then \( du = f(x) + xf'(x) \) and \( v = f(x) \), so
\[
I = \int_a^b x f(x) f'(x) \, dx
= xf(x)f'(x)|_a^b - \int_a^b f(x)[f(x) + xf'(x)] \, dx
= -\int_a^b f(x)^2 \, dx - \int_a^b xf(x)f'(x) \, dx
= -1 - I.
\]
where we used the fact that \( f \) vanishes at \( a \) and \( b \). Solving the equation \( I = -1 - I \), we obtain
\[
I = -\frac{1}{2}.
\]

Solution 2: Integration by parts with \( u = x \) and \( dv = f(x)f'(x) \, dx \) is even easier.
6. (20 + 15 points) Assume $h : \mathbb{R} \to \mathbb{R}$ is differentiable and

$$1 \leq h'(x) \leq 2,$$

for all $x \in \mathbb{R}$.

(a) Show that $h$ is a diffeomorphism, i.e., $h$ is a bijection and its inverse $h^{-1}$ is differentiable.

(b) Show that $h$ maps $F_\sigma$-sets to $F_\sigma$-sets. That is, for every $F_\sigma$-set $A \subset \mathbb{R}$, prove that $h(A)$ is an $F_\sigma$-set.

(c) (5 extra credit points) Show that $E \subset \mathbb{R}$ is measurable if and only if $h(E)$ is measurable.

(d) (10 extra credit points) Show that for every measurable set $E$,

$$m(h(E)) \leq 2m(E).$$

Solution: (a) Since $h' \geq 1 > 0$, $h$ is strictly increasing, hence 1–1. Integrating $1 \leq h' \leq 2$, we obtain

$$x + a \leq h(x) \leq 2x + a,$$

where $a = h(0)$. This implies $h(x) \to \pm \infty$, as $x \to \pm \infty$, so $h$ is also onto. Therefore, $h$ is a bijection. Since $h'$ is never zero, $h^{-1}$ is differentiable everywhere, by the theorem on differentiability of the inverse function (and $(h^{-1})'(y) = 1/h'(x)$, where $y = h(x)$). Thus $h : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism.

(b) First let us show that $h$ takes closed sets to closed sets. Assume $F \subset \mathbb{R}$ is closed. Then its complement $F^c$ is open, so $h(F^c) = (h^{-1})^{-1}(F^c)$ is open, by continuity of $h^{-1}$ (the pre-image of an open set is open). But $h(F^c) = h(F)^c$, so $h(F)^c$ is open, implying that $h(F)$ is closed. This proves our assertion.

Now let $A$ be an $F_\sigma$-set. Then $A$ can be written as

$$A = \bigcup_{n=1}^\infty F_n,$$

where $F_n$ is closed for each $n$. Therefore, $h(F_n)$ is closed, so

$$h(A) = \bigcup_{n=1}^\infty h(F_n)$$

is an $F_\sigma$-set.

(c) Assume $E$ is measurable. Then by the regularity of Lebesgue measure, $E = A \cup Z$, where $A$ is a $G_\delta$- or an $F_\sigma$-set and $Z$ is a zero set. It follows from part (d) (or from
problem 1 on the last year’s final exam) that \( h \) takes zero sets to zero sets, so \( h(Z) \) is a zero set. A completely analogous argument to the one in (b) shows that \( h \) takes \( G_δ \)-sets to \( G_δ \)-sets. Thus \( h(A) \) is a \( G_δ \) or an \( F_σ \)-set. Therefore, \( h(E) = h(A) \cup h(Z) \) is measurable as a union of two measurable sets.

(d) We are led by the following principle: if \( h \) stretches/shrinks intervals by a factor of at most \( λ \), then it stretches/shrinks any measurable set by a factor of at most \( λ \).

Formally, recall that for a measurable set \( E \), \( m(E) \) is just the outer measure \( m^*(E) \). If \( E \) is unbounded, then so is \( h(E) \), in which case they both have infinite measure, and \( ∞ ≤ 2∞ \).

So assume that \( E \) is bounded and hence \( m(E) < ∞ \). Let \( ε > 0 \) be arbitrary. Then there exists a covering \( \{I_n\} \) of \( E \) by open intervals such that its total length satisfies

\[
\sum_{n=1}^{∞} |I_n| < m(E) + \frac{ε}{2}.
\]

Since \( h \) is a diffeomorphism, \( J_n = h(I_n) \) is an open interval, for each \( n \). Furthermore,

\[
h(E) \subset h \left( \bigcup_{n} I_n \right) = \bigcup_{n} h(I_n) = \bigcup_{n} J_n,
\]

so \( \{J_n\} \) is a covering of \( h(E) \). Let \( I_n = (a_n, b_n) \). By the Mean Value Theorem,

\[
|J_n| = h(b_n) - h(a_n) = h'(c_n)(b_n - a_n) \leq 2 |I_n|
\]

so \( ∑ |J_n| \leq 2 ∑ |I_n| \). By the definition of outer measure,

\[
m(h(E)) \leq ∑_{n=1}^{∞} |J_n| \leq 2 ∑_{n=1}^{∞} |I_n| < 2m(E) + ε.
\]

Since \( ε > 0 \) was arbitrary, \( m(h(E)) \leq 2m(E) \).
...a measurably excellent summer break!