These are notes from the February 20, 2008 meeting with Dr. Jeffrey Scargle from NASA-Ames. Let us suppose that we have a causal set \((C, \prec)\) representing space-time. Our goal is to use the Feynman path integral (or rather path sum) approach to compute probability amplitudes of events such as a photon traveling from a light source and hitting a detector (e.g., a telescope) with a certain velocity. In the Feynman approach to quantum mechanics, the probability amplitude (also known as the propagator) associated with a path \(\gamma\) is a complex number of the form

\[
A = \exp\{iS(\gamma, t)\},
\]

where \(t\) is time-length of the path and \(S(\gamma, t)\) is the so called action\(^1\). Classically, \(S(\gamma, t)\) is the integral from 0 to \(t\) of the difference between the kinetic and potential energy. The main question for us is: what should this term be for photons and in the context of causal sets?

One of the main problems is that in quantum mechanics time is given (like in Newtonian mechanics) and not dynamic like in general theory of relativity. So we need to replace this absolute time \(t\) by something else. Dr. Scargle postulated that \(t\) should be replaced by so called proper time \(\tau\), and that the probability amplitude for photons in our context should be given by

\[
A = \exp\left(\frac{iE}{\hbar}\tau\right).
\]

where \(E\) is the energy of the photon and \(\hbar = \hbar/2\pi\) is the Planck constant divided by \(2\pi\) (so \(\hbar = 6.58 \times 10^{-16}\) eV.sec). The energy \(E\) is of the order of magnitude of tera electron volts (TeV), or \(10^{12}\) eV.

What is proper time? It is an invariant of the observer given by

\[
c^2\tau^2 = c^2t^2 - x^2 - y^2 - z^2,
\]

where \((x, y, z, t)\) are the coordinates of our point in space-time and \(c\), as usual, is the speed of light. Observe that the expression on the right-hand side plays the same role in Minkowski geometry as arclength in Euclidean geometry. Or, to quote Wikipedia: “In relativity, proper time is time measured by a single clock between events that occur at the same place as the clock. It depends not only on the events but also on the motion of the clock between the events. An accelerated clock will measure a shorter proper time between two events than a non-accelerated (inertial) clock between the same events. The twins paradox is an example of this.”

What is the order of magnitude of \(\tau\)? It is so called Planck time, which is the time it would take a photon travelling at the speed of light in a vacuum to cross a distance equal to the Planck length, which is about \(10^{-20}\) times the diameter of a proton, or \(1.6 \times 10^{-35}\) meters. Thus \(\tau\) is of the order of \(10^{-44}\) seconds. So

\[
\frac{E}{\hbar}\tau \approx \frac{10^{12}}{10^{-16}} \times 10^{-44} = 10^{-16}.
\]

\(^{1}\)Note that the notation differs slightly from the January 30 notes.
Now suppose that we have a light source located at some point \((x_1, t_1)\) in our causal set, which for now we assume is 1+1 dimensional (i.e., only one spatial dimension with coordinate \(x\)). The photon leaves the light source, travels for some time and is absorbed by a detector, such as a telescope. How do we represent this detector as a subset of our causal set? By the way, since the distances between points in our causal set have no physical meaning, we may as well assume that \(C\) is just the integer lattice \(\mathbb{Z}^2\) in \(\mathbb{R}^2\).

Let us normalize \(c\) to be one, and switch to **light-cone coordinates**

\[
x_+ = x + t, \quad x_- = -x + t.
\]

Note that the second formula differs from the one used in class; the reason is that we want the light cone to be mapped to the first quadrant in the new coordinates system. If we set \(x_- = x - t\), then the light cone does not go to the first quadrant.

This is a linear change of coordinates with matrix

\[
\begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix} = \sqrt{2} \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{bmatrix},
\]

so it corresponds to a clockwise rotation by \(\theta = \pi/4\) followed by scaling by a factor of \(\sqrt{2}\). [By the way: does such a nice coordinate change exist in 3+1D space time?]

In the light-cone coordinates, the light cone based at the origin \((0, 0)\) is just the first quadrant \(x_+ \geq 0, x_- \geq 0\). Since our photon travels at the speed of light \(c\) and for us \(c = 1\), its classical path (in the old \((x, t)\) coordinates) is given by the equation \(x = ct = t\), i.e., \(x_- = 0\). In other words, the photon travels along the \(x_+\)-axis in the light-cone coordinate system. But Feynman (in fact, Nature) tells the photon to explore all paths, so we are faced with the question: *do all paths make equal contributions*, or can we somehow separate the most significant paths and ignore the rest in our calculation? One idea is to divide the paths according to the number of steps they are composed of. Intuitively, shorter paths are more important than longer ones. This question needs to be further explored.

Now let us go back to the question of how to represent the detector \(D\) as a subset of our causal set \(C\). It is clear that \(D\) should consist of more than one point. It makes sense to describe \(D\) as a set of the form \(x > x_D\) (in the old space-time coordinates \((x, t)\)) or even a strip \(a < x < b\), where \(t\) can be anything. Let us look at the first possibility. In the light-cone coordinates \(x > x_D\) corresponds to the region \(x_+ - x_- > 2x_D\). To see this we can use the change of coordinates \((x_+, x_-) \mapsto (x, t)\) which is easily seen to be

\[
x = \frac{x_+ - x_-}{2}, \quad t = \frac{x_+ + x_-}{2}.
\]

Observe also that in the light-cone coordinates proper time \(\tau\) is given by

\[
\tau^2 = t^2 - x^2 = \frac{(x_+ + x_-)^2}{4} - \frac{(x_+ - x_-)^2}{4} = x_+ x_-,
\]

so \(\tau = \sqrt{x_+ x_-}\) or \(\Delta \tau = \sqrt{\Delta x_+ \Delta x_-}\).

Another fundamental question is: **with what probability is the photon going to have velocity \(v\) at the moment of absorption?** To figure this out, we need to find the part of \(D\) which a photon with velocity \(v\) can reach in time \(t\). Since

\[
v = \frac{\Delta x}{\Delta \tau},
\]

assuming that our light source is located at the origin and the point of space-time where it is absorbed has coordinates \((x_+, x_-)\), we obtain

\[
v = \frac{x_+ - x_-}{x_+ + x_-}.
\]
So the region of \( D \) hit by photons with velocity \( v \) is

\[
D_v = \{(x_+, x_-) : x_- = fx_+, x_+ - x_- > 2x_D\}, \quad \text{where} \quad f = \frac{1 - v}{1 + v}.
\]

Observe that \( D_v \) lies on a straight line with positive slope \( 0 \leq f \leq 1 \) in the \((x_+, x_-)\)-plane. See Fig. 1.

**Figure 1.** The detector \( D \) and the set of points \( D_v \) with velocity \( v \)

It is likely that it makes more sense to look at the region of \( D \) hit by photons whose velocities are within a certain range, e.g., \( v \in V \), where \( V \) is some interval. Then the corresponding region of \( D \) would be the union of \( D_v \), for \( v \in V \), which is a cone. It is then natural to ask:

What is the probability that a photon with velocity \( v \in V \) will be absorbed by the detector as a function of its energy \( E \) and velocity \( v \)?