**Exercise 2.2.6:** Let $x$ be a fixed point of a map $f$ on the real line such that $|f'(x)| = 1$ and $f''(x) \neq 0$. Show that arbitrarily close to $x$ there is a point $y$ such that the iterates of $y$ do not converge to $x$.

**Proof:** Suppose for instance that $f'(x) = 1$ and $f''(x) > 0$. By the Taylor formula,

$$f(y) = f(x) + f'(x)(y - x) + f''(x)(y - x)^2 + R(y)$$

where $R(y)/(y - x)^2 \to 0$, as $y \to x$. Therefore, there exists a neighborhood $U$ of $x$ such that for every $y \in U$, $|R(y)| < f''(x)(y - x)^2 / 2$. It follows that for all $y \in U$, we have $f(y) > y$, so if $y > x$, then $f(y) > y > x$. By induction, as long as $f^n(y) \in U$, we have $f^{n+1}(y) > f^n(y)$, i.e., the iterates of $y$ move away from $x$ and therefore cannot converge to $x$.

The other cases can be dealt with analogously. The problem can also be solved graphically using cobweb diagrams, but this solution is not rigorous.

**Problem 2.2.13:** Suppose that $I$ is a closed bounded interval and $f : I \to I$ is such that $d(f(x), f(y)) < d(x, y)$ for any $x \neq y$ (this is weaker than the assumption of the Contraction Principle). Prove that $f$ has a unique fixed point $x_0 \in I$ and that $\lim_{n \to \infty} f^n(x) = x_0$ for any $x \in I$.

**Proof:** The function $f$ is Lipschitz, hence continuous and continuous maps of $I$ have a fixed point. Alternatively, let $m = \inf\{|f(x) - x| : x \in I\}$. Since $f$ is continuous and $I$ is compact, $m$ is achieved at some point $x_0 \in I$: $m = |f(x_0) - x_0|$. Suppose that $m > 0$, i.e., $f(x_0) \neq x_0$ and let $y_0 = f(x_0)$. Then

$$|f(y_0) - f(x_0)| = |y_0 - x_0| = m,$$

which is a contradiction. Therefore, $f(x_0) = x_0$, that is, $x_0$ is a fixed point of $f$. If $x$ is another fixed point of $f$, then necessarily $x = x_0$, otherwise

$$|x - x_0| = |f(x) - f(x_0)| < |x - x_0|,$$

which is clearly impossible.

Let $x \in I$ be arbitrary and define $x_n = f^n(x)$. Denote by $A$ the set of all accumulation (or cluster) points of the sequence $(x_n)$. This means that $y \in A$ if there exists a subsequence of $(x_n)$ converging to $y$.

Since

$$|x_{n+1} - x_0| = |f(x_n) - f(x_0)| < |x_n - x_0|,$$

the sequence $|x_n - x_0|$ is strictly decreasing and bounded below, hence convergent, say to $\delta$. Therefore, $A$ is contained in the set $\{x_0 - \delta, x_0 + \delta\}$. If $\delta = 0$, $x_n \to x_0$ and we are done. So suppose $\delta > 0$. Let $y \in A$ be arbitrary. It is not hard to see that $f(y) \in A$, so on one hand $|y - x_0| = \delta$, but on the other hand,

$$|f(y) - x_0| = |f(y) - f(x_0)| < |y - x_0| = \delta,$$

which is a contradiction. Therefore, $\delta = 0$ and $x_n \to x_0$.  

\[\square\]
Problem 2.2.14: Show that the assertion of the previous exercise is not valid for $I = \mathbb{R}$ by constructing a map $f : \mathbb{R} \to \mathbb{R}$ such that $d(f(x), f(y)) < d(x, y)$ for $x \neq y$, $f$ has no fixed point, and $d(f^n(x), f^n(y))$ does not converge to zero for some $x, y$.

Solution: Let $f(x) = \pi + x - \arctan x$. Observe that
\[
0 \leq f'(x) = 1 - \frac{1}{1 + x^2} < 1.
\]
Therefore, for all $x, y \in \mathbb{R}$, $x \neq y$, there exists $c$ between $x$ and $y$ such that $f(x) - f(y) = f'(c)(x - y)$. This implies
\[
|f(x) - f(y)| < |x - y|.
\]
On the other hand, if $f(x) = x$, then $\arctan x = \pi$, which is impossible, so $f$ does not have a fixed point.

Finally, let $x \in \mathbb{R}$ be arbitrary and let $y = f(x)$. Since $f(t) - t = \pi - \arctan t > \pi/2$, it follows that
\[
|f^n(y) - f^n(x)| = |f(f^n(x)) - f^n(x)| > \pi/2,
\]
for all $n$, so $|f^n(x) - f^n(y)| \not\to 0$, as $n \to \infty$. \qed