Exercise 6.1.1: Suppose that $f$ is an orientation and length-preserving homeomorphism of $S^1$. Let $F : \mathbb{R} \to \mathbb{R}$ be a lift of $f$. Then $F$ also preserves length, i.e., $F$ is a rigid motion of $\mathbb{R}$, so $F$ has to be either a translation of a reflection. Since $f$ is orientation preserving, $F$ is strictly increasing, so it must be a translation, e.g. $F(x) = x + \alpha$. This implies that $f$ is a rotation by $\alpha$. □

Exercise 6.1.2: Suppose that $p$ is an attracting fixed point for a $C^1$ diffeomorphism $f : X \to X$, where $X$ is a smooth manifold, or just a nice subset of a Euclidean space (so that “volume” makes sense in it). Let $r > 0$ be sufficiently small so that the ball $B_{3r}$ of radius $3r$ centered at $p$ is contained in the basin of attraction of $p$ (i.e., for every $x \in B_{3r}$, $f^n(x) \to p$, as $n \to \infty$). We claim that there exists $N > 0$ such that $f^N(B_{2r}) \subset B_r$, where $B_r$ is the $r$-ball at $p$.

Since $p$ is stable, there exists an open set $W \subset B_{r}$ such that $f^n(W) \subset B_r$, for all $n \geq 0$.

Now consider an arbitrary $x \in B_{2r}$ (the closed 2$r$-ball). Since $f^n(x) \to p$, there exists $N_x \in \mathbb{N}$ such that for all $n \geq N_x$, $f^n(x) \in W$. Let

$$V_x = f^{-N_x}(W) \cap B_{2r}.$$ 

Then $V_x$ is an open set in $B_{2r}$ and the collection $\{V_x : x \in B_{2r}\}$ clearly covers $B_{2r}$. Since $B_{2r}$ is compact, there exists a finite subcover $\{V_{x_1}, \ldots, V_{x_k}\}$ of $B_{2r}$. Let

$$N = \max(N_{x_1}, \ldots, N_{x_k}).$$

We claim that $f^N(B_{2r}) \subset B_r$. Let $x \in B_{2r}$ be arbitrary. Then $x \in V_{x_i}$, for some $1 \leq i \leq k$, so $f^{N_{x_i}}(x) \in W$. Therefore,

$$f^N(x) = f^{N-N_{x_i}}(f^{N_{x_i}}(x)) \in f^{N-N_{x_i}}(W) \subset B_r,$$

proving the above claim.

It follows that $\text{vol}(f^N(B_{2r})) \leq \text{vol}(B_r) < \text{vol}(B_{2r})$, so $f^N$ is not volume preserving. Therefore, $f$ is not volume preserving (otherwise, $f^N$ would be), as claimed. □

Exercise 6.1.4: The divergence of the vector field is zero, so its flow is area-preserving. Observe that the origin is a center and that all non-trivial orbits are concentric circles centered at the origin. The equations model a non-damped harmonic oscillator. □

Exercise 6.1.5: The divergence is again zero, so the flow preserves area. This is an equation of a non-damped pendulum. □

Exercise 6.1.6: The divergence is $-1$, so the flow is not area-preserving. □

Exercise 6.1.7: Suppose that $p$ is an attracting fixed point for $f$. Then there exists a neighborhood $U$ of $p$ (namely, the basin of attraction of $p$) such that for every $x \in U$, $f^n(x) \to p$,
as \( n \to \infty \), i.e., \( \omega(x) = \{p\} \). This means that for all \( x \in U \setminus \{p\} \), \( x \notin \omega(x) \), i.e., \( x \) is not recurrent. Since the only recurrent point in the open set is \( p \), it follows that the set of recurrent points is not dense. \( \square \)