## Midterm Exam

March 12, 2008

Due on March 19 at 3 PM

**Name:**

<table>
<thead>
<tr>
<th>Score</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>XC</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
</tr>
</tbody>
</table>

**Explain your work**
1. **(25 points)** Assume that $f : \mathbb{R}^k \to \mathbb{R}^k$ is a $C^1$ diffeomorphism and $p$ is a periodic point of $f$ of period three. Let $p_i = f^i(p), i = 0, 1, 2$. Show that the matrices

$$
Df^3(p_0), \ Df^3(p_1), \ Df^3(p_2)
$$

have the same spectrum.

**(Hint:** Use the chain rule.)

**Proof:**
2. (25 points) Let $X$ be a metric space and let $f : X \to X$ be a homeomorphism. A point $p \in X$ is called a **non-wandering point** of $f$ if for every neighborhood $U$ of $p$ there exists an integer $n > 0$ such that $f^n(U) \cap U$ is non-empty. The set of non-wandering points of $f$ is denoted by $\Omega(f)$.

(a) Show that $\Omega(f)$ is closed.

(b) Show that $\Omega(f)$ is invariant under $f$.

(c) Find $\Omega(f)$ for an arbitrary circle rotation $f = R_\alpha : S^1 \to S^1$.

Solution:
3. (25 points) Let $f : S^1 \to S^1$ be defined by $f(z) = z^2$.

(a) Show that periodic points of $f$ are dense in $S^1$.
(b) How many periodic points of period $n$ (for $n = 1, 2, \ldots$) does $f$ have?
(c) What is the degree of $f$?

(Hint for (a) and (b): Write $f$ in additive notation as a map of $[0, 1]$ and sketch the graph of its iterates.)

Solution:
4. (25 points) Let $f : S^1 \to S^1$ be an orientation preserving homeomorphism with finitely many fixed points. If $f$ has an attracting fixed point, show that it must have a repelling fixed point.

Proof:
**Extra credit (20 points)** For a diffeomorphism $f$ and a nonzero vector $v$, the number

$$
\chi(v) = \lim_{n \to \infty} \frac{1}{n} \log \| Df^n(v) \|
$$

is called the **Lyapunov exponent** of $v$. Compute the Lyapunov exponents of a linear isomorphism (i.e., non-singular matrix) $L : \mathbb{R}^2 \to \mathbb{R}^2$.

**Solution:**