Foliations of Three Dimensional Manifolds*

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Abstract

The theory of foliations began with a question by H. Hopf in the 1930's: "Does there exist on S^3 a completely integrable vector field?" By Frobenius' Theorem, this is equivalent to: "Does there exist on S^3 a codimension one foliation?" A decade later, G. Reeb answered affirmatively by exhibiting a C^∞ foliation of S^3 consisting of a single compact leaf homeomorphic to the 2-torus with the other leaves homeomorphic to $\mathbb{R}^2$ accumulating asymptotically on the compact leaf. Reeb's work raised the basic question: "Does every C^∞ codimension one foliation of S^3 have a compact leaf?" This was answered by S. Novikov with a much stronger statement, one of the deepest results of foliation theory: *Every C^2 codimension one foliation of a compact 3-dimensional manifold with finite fundamental group has a compact leaf.* The basic ideas leading to Novikov's Theorem are surveyed here.[1]

1 Introduction

Intuitively, a foliation is a partition of a manifold M into submanifolds A of the same dimension that stack up locally like the pages of a book. Perhaps the simplest nontrivial example is the foliation of $\mathbb{R}^3\backslash\{\mathbb{O}\}$ by concentric spheres induced by the submersion $\mathbb{R}^3\backslash\{\mathbb{O}\} \ni x \mapsto ||x|| \in \mathbb{R}$. Abstracting the essential aspects of this example motivates the following general

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[1] We follow here the general outline presented in [Ca]. For a more recent and much broader survey of foliation theory, see [To].

Definition. Let M be a C^∞ manifold of dimension m. A C^r **foliation** $\mathcal{F}$ **of** M **of dimension** $n < m$ is a C^r atlas maximal with respect to the following properties:

(i) If $(U, \varphi) \in \mathcal{F}$, then $\varphi(U) = U_1 \times U_2 \subset \mathbb{R}^n \times \mathbb{R}^{m-n}$ where U_1 and U_2 are open disks in $\mathbb{R}^n$ and $\mathbb{R}^{m-n}$ respectively.

(ii) If $(U, \varphi), (V, \psi) \in \mathcal{F}$, and if $U \cap V \neq \emptyset$, then $\psi \circ \varphi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is of the form

$$\psi \circ \varphi^{-1}(x, y) = (h_1(x, y)), h_2(y)), \quad (x, y) \in U_1 \times U_2,$$

where h_1 and $h_2 \in C^r(\mathbb{R}^m)$.

Condition (ii) guarantees that the local "stack of pages" is invariant under coordinate changes $\psi \circ \varphi^{-1}$. The **codimension** of such a foliation $\mathcal{F}$ is $m - n$. Given a foliation $\mathcal{F}$ on M, a **plaque** α **of** $\mathcal{F}$ is a subset of M of the form $\varphi^{-1}(U_1 \times \{c\})$ for some chart $(U, \varphi) \in \mathcal{F}$ and some $c \in \mathbb{R}^{m-n}$. A **path of plaques of** $\mathcal{F}$ is a finite sequence of plaques $\{\alpha_1, \ldots, \alpha_n\}$ such that $\alpha_i \cap \alpha_{i+1} \neq \emptyset$, $i = 1, \ldots, n-1$. We may define on M an equivalence relation $p \sim q$ iff there exists a path of plaques with $p \in \alpha_1$, $q \in \alpha_n$. An equivalence class of this relation is called a **leaf of** $\mathcal{F}$.

2 The Reeb foliation of S^3

We have seen that $\mathbb{R}^3$ admits a codimension one foliation the leaves of which are all compact submanifolds. Perhaps the next simplest 3-dimensional manifold to consider is S^3. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the submersion defined by $f(x_1, x_2, x_3) = \alpha(r^2)e^{x_3}$, where $r = \sqrt{x_1^2 + x_2^2}$ and $\alpha : \mathbb{R} \to \mathbb{R}$ is a C^∞ function such that $\alpha(1) = 0$, $\alpha(0) = 1$ and $\alpha'(t) < 0$ for $t > 0$. The leaves are the connected components of the submanifolds $f^{-1}(c)$, $c \in \mathbb{R}$.

The leaves in the interior of the solid cylinder $\mathcal{C} : x_1^2 + x_2^2 = 1$ are all homeomorphic to $\mathbb{R}^2$: in fact, they can be parametrized by $(x_1,\, x_2) \in D^2 \mapsto (x_1, x_2, \ln(c/\alpha(r^2)))$, where $c > 0$. The boundary $\partial\mathcal{C} : x_1^2 + x_2^2 = 1$ is a leaf, and outside $\mathcal{C}$, the leaves are also homeomorphic to cylinders.

Now take $\alpha(r^2) = \exp(-\exp(1/1 - r^2))$ and restrict f to the solid cylinder $D^2 \times \mathbb{R} \to \mathbb{R}$. The leaves are the graphs of the functions $x_3 = \exp(1/1 - r^2) + b$, $b \in \mathbb{R}$.

Restrict f to $D^2 \times [0, 1]$. For all $x, y : x^2 + y^2 \leq 1$, identify $(x, y, 0)$ and $(x, y, 1)$. This induces a foliation of $D^2 \times S^1$ whose boundary, the 2-torus, is a compact leaf. S^3 can then be obtained by gluing together two such solid tori so that the meridians of one are identified with the parallels of the other, and vice versa. This induces a foliation of S^3 of codimension one called the Reeb foliation of S^3. One leaf is compact and homeomorphic to T^2; all the other leaves are homeomorphic to $\mathbb{R}^2$ and accumulate on the compact leaf.

3 Topology of the Leaves

Let M be a manifold foliated by a C^r n-dimensional foliation $\mathcal{F}$. Each leaf F of a C^r foliation $\mathcal{F}$ of M has a C^r differential manifold structure induced by the charts (U, ϕ) of $\mathcal{F}$, called the **intrinsic structure of** F. This is defined as the maximal C^r atlas containing the set

$$\mathcal{B} = \{(\alpha, \bar{\phi}) \mid \alpha \subset F \text{ is a plaque of } U \text{ with } (U, \phi) \in \mathcal{F}\},$$

where $\bar{\phi} = \phi_1|_\alpha$ and $\phi = (\phi_1, \phi_2)$.

The plaques α of $\mathcal{F}$ such that $\alpha \subset F$ define a basis of open sets giving F the structure of an immersed submanifold of M. F may not be embedded. For example, although F may not be locally connected in the subspace topology, in the intrinsic topology it is a manifold and thus necessarily locally connected. To study the topology of the leaves further requires the following notions and results.

Definition. Let $f : M \to N$ be a C^r map and $S \subset N$ a submanifold of N. We say that f is **transverse to** S at $x \in M$ if $y = f(x) \notin S$ or if $y = f(x) \in S$ and $T_yN = T_yS + Df(x) \cdot T_xM$. When f is transverse to S at every point of M we say that f **is transverse** to S. When S^s, $M^m \subset N^n$ are two C^r submanifolds of N we say that S meets M transversely at $x \in S$ if the embedding $i : S \to N$, $i(y) = y$, is transverse to M at x. This condition is equivalent to $T_x = T_xM + T_xS$ in case $x \in S \cap M$.

Definition. Let Σ be a submanifold of M and let $\mathcal{F}$ be a foliation of M. We say that Σ is transverse to $\mathcal{F}$ when Σ is transverse to every leaf of $\mathcal{F}$ that it meets. When $\dim(\Sigma) + \dim(\mathcal{F}) = \dim(M)$ we say Σ is a **transverse section of** $\mathcal{F}$.

Given $p \in M$ we can always find a transverse section of $\mathcal{F}$ passing through p.

Theorem (Transverse Uniformity).[2] *Let F be a leaf of $\mathcal{F}$. Given $q_1, q_2 \in F$, there exist transverse sections Σ_1, Σ_2 of $\mathcal{F}$, with $q_i \in \Sigma_i$, $(i = 1, 2)$, and a C^r diffeomorphism $f : \Sigma_1 \to \Sigma_2$, such that for any leaf F' of $\mathcal{F}$, one has $f(F' \cap \Sigma_1) = F' \cap \Sigma_2$.*

[2]Proofs of all theorems stated here may be found in [Ca].

We now have the following result, which can sometimes be used to decide whether a given submanifold L' of M is a leaf of some foliation.

Corollary. *Suppose $\mathcal{F}$ is a foliation of M, F a leaf of $\mathcal{F}$, and Σ any transverse section of $\mathcal{F}$ such that $F \cap \Sigma \neq \emptyset$. Then there are exactly three mutually exclusive possibilities:*

(1) $\Sigma \cap F$ is discrete (i.e., every point of $\Sigma \cap F$ is an isolated point). In this case, F is an embedded leaf.
(2) $\overline{\Sigma \cap F}$ contains an open set. In this case, F is called a **dense** *leaf.*
(3) $\overline{\Sigma \cap F}$ is a perfect set with empty interior. In this case, F is called an **exceptional** *leaf.*

For example, we can use this to show that the topologist's sine curve S is not a leaf of any foliation of a open submanifold of $\mathbb{R}^2$. In fact, consider a transverse section Σ connecting a point $x \in S$ on the vertical segment passing through the origin with a point $y \in S$ on the arc connecting the vertical segment with $(1, 0)$. Clearly, cases (2) and (3) do not apply; but since y is isolated while x is not, neither does case (1).

4 Holonomy and the Stability Theorems

We are interested here in the behaviour of leaves near a fixed compact leaf. First, we need to recall a definition from algebraic topology:

Definition. Let M be a topological space and $p_0 \in M$. A path $\gamma : [0, 1] \to M$ that begins and ends at p_0 is called a **loop** at p_0. The corresponding set $\pi_1(M, p_0)$ of path homotopy classes $[\gamma]$ of loops based at p_0, with the operation $[\gamma_0] * [\gamma_1] = [\gamma_0 * \gamma_1]$ where $*$ concatenates loops at p_0, is called the **fundamental group** of M relative to the **base point** p_0.

Definition. Let X be a topological space and $p_0 \in X$. Consider the set of local homeomorphisms $f : V \to X$, V a neighborhood of p_0, such that $f(p_0) = p_0$. The corresponding set of germs $G(X, p_0)$ with operation defined by $[f] * [g] = [f] * [g]$ is called the **group of germs** of M at p_0.

We will be especially interested in the case where X is a small transverse section Σ_0 passing through a point p_0 of a leaf F. The next result is a consequence of Transverse Uniformity:

Theorem. *Let M be a manifold with foliation $\mathcal{F}$, $\gamma : [0, 1] \to M$ a curve in a leaf F of $\mathcal{F}$, and Σ_0, Σ_1 transverse sections to F at $\gamma(0) = p_0$, $\gamma(1) = p_1 \in F$. Then there exists a diffeomorphism $f_\gamma : \Sigma_0 \to \Sigma_1$ called the* **holonomy map associated to γ**

such that $f_\gamma(p_0) = p_1$.

Again, we will be particularly concerned with the case where $p_0 = p_1$ and $\Sigma_0 = \Sigma_1$. In this situation, for $x \in \Sigma_0$ sufficiently near p_0, $f_\gamma(x)$ represents the "first return" to Σ_0 of the leaf passing through x "over" the curve γ.

Theorem. *Let M be a manifold with foliation $\mathcal{F}$; $\gamma_i : [0,1] \to M,\ i = 0,1$ paths in a leaf F of $\mathcal{F}$ such that $\gamma_i(0) = p_0$, $\gamma_i(1) = p_1,\ i = 0,1$; Σ_0, Σ_1 transverse sections to F at p_0, p_1; f_{γ_i} be holonomy maps associated to γ_i; and $[f_{\gamma_i}]$ the germs of f_{γ_i} at p_0. Then if $p_0 = p_1$ and $\Sigma_0 = \Sigma_1$, the transformation $\gamma \mapsto [f_{\gamma^{-1}}]$ induces a homomorphism*

$$\Phi : \pi_1(F, p_0) \to G(\Sigma_0, p_0)$$

between the fundamental group of F at p_0 and the group of germs of C^r diffeomorphisms of Σ_0 which leave p_0 fixed.

Definition. The subgroup $Hol(F, p_0) = \Phi(\pi_1(F, p_0))$ of $G(\Sigma_0, p_0)$ is called the **holonomy group** of F at p_0.

As an example, we see that when the leaf F is simply connected then $Hol(F, p_0)$ is trivial: i.e., F does *not* return over any path to intersect any transverse section through p_0.

The holonomy group of F induces a local action of $\pi_1(F, p_0)$ on Σ_0, namely, $\pi_1(F, p_0) \times \Sigma_0 \to \Sigma_0$ given by $([\gamma], x) \mapsto f_\gamma(x)$. It turns out that when F is compact, this action characterizes the foliation in a neighborhood of F. The details of this claim involve the notions of conjugacy between holonomies and local equivalence of foliations. Development along these lines then leads to

Theorem (Local Stability). *Let $\mathcal{F}$ be a C^1 codimension n foliation of M and F a compact leaf with finite fundamental group. Then there exists a neighborhood of F, saturated in $\mathcal{F}$, in which all the leaves are compact with finite fundamental groups.*

Theorem (Global Stability). *Suppose $\mathcal{F}$ is a codimension one foliation of a compact, connected manifold M and F a compact leaf of $\mathcal{F}$ with finite fundamental group. Then all the leaves of $\mathcal{F}$ are compact with finite fundamental group.*

5 Novikov's Theorem

The natural question now arises: What manifolds have foliations containing a compact leaf? The answer, in part, is given by

Theorem (Novikov). *Every C^2 codimension one foliation of a compact three dimensional manifold with finite fundamental group has a compact leaf.*

To prove the theorem, one first shows the existence of a vanishing cycle, i.e., a certain map $f_0 : S^1 \to A_0$, defined below, with A_0 a leaf of M. From this the existence of *simple* vanishing cycles can be deduced, and this in turn implies that A_0 is compact.

Definition. Let A_0 be a leaf of $\mathcal{F}$. A closed curve $f_0 : S^1 \to A_0$ is called a **vanishing cycle** if f_0 extends to a differentiable map $F : [0, \varepsilon] \times S^1 \to M$ satisfying the following conditions:

(i) for every $t \in [0, \varepsilon]$, the curve $f_t : S^1 \to M$ defined by $f_t(x) = F(t, x)$ is contained in a leaf A_t of $\mathcal{F}$.

(ii) for every $x \in S^1$, the curve $t \mapsto F(t, x)$ is transverse to $\mathcal{F}$.

(iii) for $t > 0$, the curve f_t is homotopic to a point in the leaf A_t and f_0 is not homotopic to a point in A_0.

As an example, in the Reeb foliation of the solid torus $D^2 \times S^1$, any meridian of the boundary torus is a vanishing cycle.

Proposition. *Let M be a compact manifold of dimension $n \geq 3$ with finite fundamental group and $\mathcal{F}$ a C^2 codimension one foliation of M. Then $\mathcal{F}$ has a vanishing cycle.*

From here on it is assumed that M is a compact, orientable, three-dimensional manifold and $\mathcal{F}$ is a C^2 foliation of M of codimension one. Given a leaf A of $\mathcal{F}$ denote by $\hat{A}$ its universal covering and by $\pi_A : \hat{A} \to A$ the covering projection. Since f_t for $t > 0$ is homotopic to a point in A_t, f_t can be lifted to a curve $\hat{f}_t : S^1 \to \hat{A}_t$ such that $\pi_{At} \circ \hat{f}_t = f_t$.

Proposition. *There exists a sequence $t_n \to 0$ such that for all $n \geq 1$ the leaf A_{t_n} contains a vanishing cycle whose extension $F(t', x)$ has the property that the $f_{t'}(x) : S^1 \to A_{t'}$ are simple curves.*

Proposition. *If A is a noncompact leaf of a foliation $\mathcal{F}$ of codimension one of a compact manifold, then for every $p \in A$ there exists a closed curve γ passing through p that is transverse to $\mathcal{F}$.*

Theorem. *Let A_0 be a leaf of $\mathcal{F}$ which admits a vanishing cycle whose extension $F : [0, \varepsilon] \times S^1 \to M$ has the property that the $f_t(x)$ are simple curves. Then*

(i) A_0 is a compact leaf.

(ii) for every $t \in (0, \varepsilon]$, A_t *is diffeomorphic to* $\mathbb{R}^2$.

In fact, even more can be shown: the compact leaf A_0 is homeomorphic to T^2.

References

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[To] Philippe Tondeur, *Geometry of Foliations*, Birkhauser, 1997.