MATH 213, SPRING 2009
HOMEWORK 1 SOLUTIONS

1. Show that every metric space is Hausdorff.

Proof: Let (X, d) be a metric space and assume $x, y \in X$ are distinct. Then $r = d(x, y)/3$ is strictly positive. The balls $B(x, r)$ and $B(y, r)$ are open sets, they contain x and y respectively and are disjoint (otherwise, if $z \in B(x, r) \cap B(y, r)$, then $d(x, y) \leq d(x, z) + d(z, y) \leq 2r < d(x, y)$, a contradiction). Therefore, any two distinct points can be separated by open sets, so X is Hausdorff. □

2. If X is a Hausdorff topological space and $K \subset X$ is compact, show that K is closed.

Proof: Assume X is Hausdorff and $K \subset X$ compact. We will show that the complement K^c of K is open. Let $x \in K^c$ be arbitrary. Since X is Hausdorff, for each $y \in K$ there exist disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. The collection $\{V_y : y \in K\}$ clearly covers K, so by compactness there exists a finite subcover $V_{y_1}, \dots, V_{y_k}$. Define

$$U = U_{y_1} \cap \cdots \cap U_{y_k}, \qquad V = V_{y_1} \cup \cdots \cup V_{y_k}.$$

Clearly, U and V are open. We claim they are also disjoint. Suppose not, e.g., $p \in U \cap V$. Then p lies in some V_{y_j} and all U_{y_i}, so U_{y_j} and V_{y_j} are not disjoint, contrary to their construction. Since V covers K, it follows that $U \subset K^c$. Therefore, x has a neighborhood contained in K^c, which proves that K^c is open. □

3. Suppose that $f : X \to Y$ is a continuous bijection, X is compact and Y is Hausdorff. Show that f is a homeomorphism.

Proof: Let $g = f^{-1}$. It suffices to show that $g : Y \to X$ is continuous, that is, that the g-preimage of every open set in X is open in Y. Since $g^{-1}(E^c) = (g^{-1}(E))^c$, for every set $E \subset X$, this is equivalent to showing that the g-preimage of every closed set is closed. Let $F \subset X$ be closed.

Lemma. *A closed subset of a compact space is itself compact.*

Proof. Suppose X is compact and $F \subset X$ is closed. Let $\mathscr{U} = \{U_i : i \in I\}$ be an open cover of F. Then $\mathscr{U}' = \mathscr{U} \cup \{F^c\}$ is an open cover of X. By compactness of X there exists a finite subcover $\{U_{i_1}, \dots, U_{i_n}, F^c\}$ of X. Then $\{U_{i_1}, \dots, U_{i_n}\}$ covers F, so $\mathscr{U}$ has a finite subcover. This proves that F is compact. □

By the Lemma, F is compact, so $g^{-1}(F) = f(F)$ is compact, by continuity of f. Since Y is Hausdorff, $f(F)$ is closed, by exercise 2. This completes the proof. □

4. Let D be the unit disk in $\mathbb{R}^2$ and let M be the space obtained by identifying the diametrically opposite points on the boundary of D (i.e., for each p on the boundary of D, identify p and $-p$). Sketch a proof that M is homeomorphic to the projective plane $P^2(\mathbb{R})$.

Proof: Since the upper unit hemisphere H in $\mathbb{R}^3$ meets every straight line through the origin, $P^2(\mathbb{R})$ is homeomorphic to $H/\sim$, where $p \sim q$ iff p and q are on the boundary of H and $q = -p$. On the other hand, H is homeomorphic to the closed unit disk D in $\mathbb{R}^2$ (a homeomorphism being the projection onto the xy-plane). It follows that $H/\sim$, hence $P^2(\mathbb{R})$, is homeomorphic to $D/\sim$, as claimed. (A rigorous proof of this claim is much more involved.) $\square$