MATH 213, SPRING 2009
HOMEWORK 10 SOLUTIONS

Chapter IV, ex. 7.4: Using additivity of the Lie bracket in each argument and exercise 7.2 in Chapter IV (see Homework 9), we obtain:

$$[X,Y] = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}, \quad [X,Z] = -\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad [Y,Z] = -\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

Chapter V, ex. 2.4: (a) Recall that

$$\operatorname{rank}\Phi = \dim V - \dim W,$$

where $W = \{w \in V : \Phi(v,w) = 0, \forall v \in V\}$ (this is the kernel of Φ). Suppose $w \in W$. Then $\Phi(v,w) = 0$, for all $v \in V$, hence $\Phi(w,w) = 0$. By positive definiteness of Φ, it follows that $w = 0$. Therefore $W = \{0\}$, so $\operatorname{rank}\Phi = \dim V$.

(b) For each $v \in V$, define $f_v : V \to \mathbb{R}$ by

$$f_v(w) = \Phi(v,w),$$

for all $w \in V$. It is clear that f_v is linear, hence in V^*. Define $L : V \to V^*$ by $L(v) = f_v$. Since Φ is bilinear, L is linear. Suppose $L(v) = 0$. Then $\Phi(v,w) = 0$, for all $w = 0$, which implies that $v = 0$. Thus L is injective. To show that L is surjective, let $f \in V^*$ be arbitrary. If $f = 0$, then $f = L(0)$. Otherwise, the kernel K of f is linear subspace of V a codimension one. Let u be a unit vector orthogonal to K and set $v = f(u)u$. We claim that $f = f_v$. If $w \in K$, then $f(w) = 0 = f_v(w)$. If $w = \lambda u$, then

$$f_v(w) = \Phi(v, \lambda u) = \lambda f(u)\Phi(u,u) = \lambda f(u) = f(\lambda u) = f(w).$$

This means that f and f_v agree on two complementary subspaces of V, thus they agree on V, by linearity. Therefore, $f = L(v)$, so L is onto, and hence a linear isomorphism between V and V^*.

(c) This is the Cauchy-Schwartz inequality. Let $v, w \in V$ be arbitrary and define

$$\phi(t) = \Phi(v + tw, v + tw).$$

Clearly, $\phi(t) \geq 0$, for all $t \in \mathbb{R}$. By bilinearity of Φ,

$$\phi(t) = \Phi(v,v) + 2t\Phi(v,w) + t^2\Phi(w,w),$$

for all t. Thus ϕ is a non-negative quadratic function of t, which is possible only if the discriminant satisfies

$$[2\Phi(v,w)]^2 \leq 4\Phi(v,v)\Phi(w,w).$$

This yields the desired inequality. □

Chapter V, ex. 2.9: I like to use the notation A^T for the transpose of A.

That $\Phi(A,B) = \operatorname{trace}(A^T B)$ is bilinear follows from linearity of matrix multiplication. Recall that for any two matrices $P, Q \in \mathscr{M}_n(\mathbb{R})$, $\operatorname{trace}(PQ) = \operatorname{trace}(QP)$. This implies that Φ is symmetric.

We have:

$$\begin{aligned}\Phi(A,A) &= \operatorname{trace}(A^T A)\\ &= \sum_{i=1}^{n}(A^T A)_{ii}\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}(A^T)_{ij}(A)_{ji}\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ji}^2\end{aligned}$$

Suppose $\Phi(A,A)=0$. Then $a_{ij}=0$, for all $1\le i,j\le n$, hence $A=0$. Therefore, Φ is positive definite. □

Chapter V, ex. 3.1: Spherical coordinates on $U=S^2\setminus\{N,S\}$ are given by $\varphi(x,y,z)=(\phi,\theta)$, where

$$\begin{aligned}x &= \sin\phi\cos\theta\\ y &= \sin\phi\sin\theta\\ z &= \cos\phi.\end{aligned}$$

Denote by $E_1=E_\phi, E_2=E_\theta$ the corresponding frame field on U. Then

$$\begin{aligned}E_\phi &= \cos\phi\cos\theta\frac{\partial}{\partial x}+\cos\phi\sin\theta\frac{\partial}{\partial y}-\sin\phi\frac{\partial}{\partial z}\\ E_\theta &= -\sin\phi\sin\theta\frac{\partial}{\partial x}+\sin\phi\cos\theta\frac{\partial}{\partial y}.\end{aligned}$$

Therefore,

$$\begin{aligned}g_{11} &= E_1\cdot E_1 = 1,\\ g_{12} = g_{21} &= E_1\cdot E_2 = 0,\\ g_{22} &= E_2\cdot E_2 = \sin^2\phi.\end{aligned}$$

In classical notation, $ds^2=d\phi^2+\sin^2\phi d\theta^2$. Observe that this is just the first fundamental form of S^2. □