MATH 213, SPRING 2009
HOMEWORK 11 SOLUTIONS

Chapter V, ex. 4.3: If V is an inner product space and $u, v \in V$, $u \neq 0$, define

$$\pi_u v = \frac{\langle u, v\rangle}{\|u\|^2} u.$$

This is the orthogonal projection of v to the line spanned by u.

For an arbitrary $p \in M$, let U_p be a coordinate neighborhood of p with a corresponding (possibly non-orthonormal) frame $\{E_1, \ldots, E_n\}$. Applying the Gram-Schmidt process to this frame pointwise, i.e., in each space T_qM ($q \in M$) with the corresponding inner product given by the Riemannian metric, we obtain an orthonormal frame $\{F_1, \ldots, F_n\}$, where

$$U_1 = E_1, \qquad F_1 = \frac{U_1}{\|U_1\|}$$

$$U_{k+1} = U_k - \sum_{i=1}^{k-1} \pi_{U_i} E_k, \qquad F_k = \frac{U_k}{\|U_k\|}.$$

Since the Riemannian metric is C^∞ and each F_k is a C^∞ function of $E_1, \ldots, E_{k-1}$, the orthonormal frame is C^∞. □

Chapter V, ex. 4.8: Let X be a continuous vector field on a C^∞ Riemannian manifold M and let $\varepsilon > 0$ be arbitrary. We will show that there exists a C^∞ vector field Y on M such that

$$\|Y(p) - X(p)\|_p < \varepsilon,$$

for every $p \in M$, where $\|\cdot\|_p$ denotes the norm on T_pM defined by the Riemannian metric of M.

We first prove the statement is true if M is covered by a single coordinate neighborhood. Then, there exists a C^∞ frame field $E_1, \ldots, E_n$ defined over the whole manifold in which X can be expressed as

$$X = \sum_{i=1}^{n} a_i E_i,$$

where a_i is a continuous real-valued function on M, for all $1 \leq i \leq n$. Without loss each E_i has norm one. For every i, there exists a C^∞ function $b_i : M \to \mathbb{R}$ such that $|b_i - a_i| < \varepsilon/n$ (Theorem V.4.8). If

$$Y = \sum_{i=1}^{n} b_i E_i,$$

then Y is C^∞ and

$$\|Y - X\| \leq \sum_{i=1}^{n} |b_i - a_i| \|E_i\| < \sum_{i=1}^{n} \varepsilon/n = \varepsilon,$$

proving the claim for manifolds covered by a single chart.

If M is not covered by a single chart, let $\{U_i, V_i, \varphi_i\}$ be a regular covering of M (cf., Ch. V.4). Let $\{f_i\}$ a C^∞ partition of unity subordinate to the covering $\{U_i\}$ such that $f_i = 1$ on V_i (recall that the closure of V_i is contained in U_i). Pick a C^∞ vector field Y_i on V_i such that $\|Y_i - X_i\| < \varepsilon$

on V_i, where $X_i = X \restriction_{V_i}$. Since $f_i Y_i$ is zero outside U_i, we can extend it to a C^∞ vector field on M and define

$$Y = \sum_i f_i Y_i.$$

It is clear that Y is a C^∞ vector field on M. Let $p \in M$ be arbitrary. Then $p \in V_i$, for some i, so since $\sum_j f_j = 1$, $Y(p) = f_i(p) Y_i(p) = Y_i(p)$. Therefore, $\|Y(p) - X(p)\| = \|Y_i(p) - X_i(p)\| < \varepsilon$, as desired. □

Chapter V, ex. 7.2: Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the unique linear map such that $L(e_i) = v_i$, $1 \le i \le 3$. Then the determinant of L is exactly $\det[x_i^j]$. If P is the parallelepiped defined by the vectors v_1, v_2, v_3, then $P = L(C)$, where C is the unit cube defined by e_1, e_2, e_3. By the change of variables formula in the Riemann integral, with $y = L(x)$ $(x \in C)$, we have:

$$\begin{aligned} \mathrm{vol}(P) &= \int_P dy \\ &= \int_{L(C)} dy \\ &= \int_C |\det L| \, dx \\ &= |\det L|. \end{aligned}$$

Note that the volume of P is actually the absolute value of $\det L$, not $\det L$. The determinant of L is the *signed* volume of P, which is positive if L preserves orientation and negative otherwise. We can also think of P as a 3-vector $v_1 \wedge v_2 \wedge v_3$ and $\|v_1 \wedge v_2 \wedge v_3\| = |\det L|$. □

Chapter VII, ex. 2.4: Since $\|Z(t)\|^2 = Z(t) \cdot Z(t) \equiv 1$ (where $\cdot$ denote the dot product in $\mathbb{R}^n$), differentiating we obtain

$$0 = 2 \frac{dZ}{dt} \cdot Z(t).$$

Since $Z(t)$ is orthogonal to TM, it follows that dZ/dt is tangent to M. Note that the problem was stated incorrectly, as DZ/dt is by definition tangent to M (which makes me wonder why I assigned this problem). □