MATH 213, SPRING 2009
HOMEWORK 2 SOLUTIONS

1. (Chapter I, ex. 3.4) Let X be a topological space and define an equivalence relation $\sim$ on X by:

   $x \sim y$ if and only if there is a connected set containing both x and y.

   The equivalence classes of $\sim$ are called the connected components of X.
   (a) Show that the connected components of a (topological) manifold are open sets.
   (b) Show that a manifold can have at most countably many connected components.

   Proof. (a) Let C be a connected component of a manifold M and let $x \in C$ be arbitrary. Since M is locally Euclidean, there exists a neighborhood U_x of x homeomorphic to an open ball in $\mathbb{R}^n$, where $n = \dim M$. Since balls are connected, so is U_x. Thus $y \sim x$, for every $y \in U_x$. Therefore, $U_x \subset C$. It follows that

   $$C = \bigcup_{x \in C} U_x,$$

   so C is open.

   (It is not true in general that components of a topological space are open. However, if X is locally connected, then the components of X *are* open. A topological space is called locally connected if for every point x and every neighborhood U of x there exists a connected neighborhood V of x such that $V \subset U$. In other words, X is locally connected if every point has a basis of connected (open, of course) neighborhoods.)

   (b) Let us now show that M has only countably many components. Denote by $\mathscr{B}$ a countable basis for the topology of M. Let $\mathscr{C}$ be the set of components of M. For each component $C \in \mathscr{C}$, choose a basis element $B = \beta(C) \in \mathscr{B}$ contained in C. (Formally, this construction requires the Axiom of Choice, which is OK.) If $C_1 \neq C_2$ are two distinct components of M, then $\beta(C_1) \cap \beta(C_2) \subset C_1 \cap C_2 = \emptyset$, so in particular, $\beta(C_1) \neq \beta(C_2)$. Therefore, $\beta : \mathscr{C} \to \mathscr{B}$ is one-to-one. Since $\mathscr{B}$ is countable, so is $\mathscr{C}$.[1] □

   □

2. Chapter II, ex. 1.5: Show that the inclusions $C^1(U) \supset C^2(U) \supset \cdots \supset C^\infty(U)$ are proper.

   Proof. Let $f_r : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^r$, where $r > 0$. Let n be an arbitrary natural number. If $n < r < n + 1$, then

   $$f^{(n)}(x) = r(r-1)\cdots(r-n+1)x^{r-n},$$

   so $f_r \in C^n(\mathbb{R})$ but $f_r \notin C^k(\mathbb{R})$, for any $k \geq n + 1$. This shows that all the inclusions above are proper. □

3. Chapter II, ex. 1.6: Prove that $C^\infty \supset C^\omega$ and that the inclusion is proper.

   Proof. If $f \in C^\omega(U)$ (i.e., if it its Taylor series expansion around any point converges to f), then it is well known that f has derivatives of all orders, hence $f \in C^\omega(U)$. To show that the inclusion is proper, let

   $$h(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

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[1] It is an exercise in elementary set theory to show that if $f : A \to B$ is 1–1 and B is countable, then so is A.

Lemma. *h is infinitely differentiable everywhere and $h^{(n)}(0) = 0$, for all $n \geq 0$.*

Proof. First of all, on $\mathbb{R} \setminus \{0\}$, h is composed of two infinitely differentiable functions, hence it is infinitely differentiable itself. At $x = 0$, however, we need to use the definition of the derivative to compute $h^{(n)}(0)$.

We claim that for each $n = 1, 2, 3, \ldots$, there exists a polynomial $Q_n(t)$ such that for all $x \neq 0$,

$$h^{(n)}(x) = e^{-1/x^2} Q_n\left(\frac{1}{x}\right). \tag{1}$$

The proof goes by induction. For $n = 1$, we have

$$h'(x) = e^{-1/x^2} \frac{2}{x^3},$$

so (1) is true, with $Q_1(t) = 2t^3$. Assume (1) is true for some $n \geq 1$. Then

$$\begin{aligned} h^{(n+1)}(x) &= \left(h^{(n)}(x)\right)' \\ &= \left(e^{-1/x^2} Q_n\left(\frac{1}{x}\right)\right)' \\ &= e^{-1/x^2} \left\{ \frac{2}{x^3} Q_n\left(\frac{1}{x}\right) - \frac{1}{x^2} Q_n'\left(\frac{1}{x}\right) \right\}. \end{aligned}$$

Therefore, if we take $Q_{n+1}(t) = 2t^3 Q_n(t) - t^2 Q_n'(t)$, we obtain $h^{(n+1)}(x) = e^{-1/x^2} Q_{n+1}\left(\frac{1}{x}\right)$, and the claim follows.

Next, we show that $h(x)/x^n \longrightarrow 0$, as $x \longrightarrow 0$ ($n \in \mathbb{N}$). That is, h is *infinitely flat at* 0 (it's flatter at 0 than any monomial x^n). We will use the fact that $t^k/e^{t^2} \longrightarrow 0$, as $t \longrightarrow \infty$, for all $k \geq 0$. This follows from L'Hospital's rule. Thus

$$\begin{aligned} \frac{h(x)}{x^n} &= \frac{e^{-1/x^2}}{x^n} \\ &= \frac{e^{-t^2}}{t^{-n}}, \qquad \text{where } x = 1/t \\ &= \frac{t^n}{e^{t^2}} \\ &\longrightarrow 0, \end{aligned}$$

as $x \longrightarrow 0$ (hence $t \longrightarrow \infty$).

Let us now prove that $h^{(n)}(0) = 0$, for all $n \geq 0$. We again use induction. For $n = 0$, this is true by definition. Assume $h^{(n)}(0) = 0$, for some $n \geq 0$. Then, using (1) and assuming $Q_n(t) = \sum_k a_k t^k$, we obtain

$$\begin{aligned} \frac{h^{(n)}(x) - h^{(n)}(0)}{x} &= \frac{e^{-1/x^2} Q_n\left(\frac{1}{x}\right)}{x} \\ &= \frac{1}{x} \sum_k a_k e^{-1/x^2} \frac{1}{x^k} \\ &= \sum_k a_k \frac{e^{-1/x^2}}{x^{k+1}}. \end{aligned}$$

We just saw that all the terms in this (finite) sum go to zero, as $x \longrightarrow 0$. Therefore, the sum goes to zero as well, and by definition of the derivative, $h^{(n+1)}(0) = 0$. By the principle of mathematical induction, $h^{(n)}(0) = 0$, for all $n \geq 0$. $\square$

The Taylor series of h at zero equals

$$\sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} x^n \equiv 0,$$

for all $x \in \mathbb{R}$. Therefore, the series does *not* converge to $h(x)$ for $x \neq 0$, hence h is not analytic. ☐

4. Chapter II, ex. 2.4

Proof. Recall that the rank of a matrix T of size $m \times n$ is the dimension of the range $T(\mathbb{R}^n)$ of T considered as a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.

Let A, B be matrices of size $m \times k$ and $k \times n$. Thinking of $A : \mathbb{R}^k \to \mathbb{R}^m$ and $B : \mathbb{R}^n \to \mathbb{R}^k$ as linear transformations, we observe that $AB(\mathbb{R}^n) \subset A(\mathbb{R}^k)$ and $\dim AB(\mathbb{R}^n) \leq \dim B(\mathbb{R}^n)$. (The latter holds because for any linear transformation $T : V \to W$ and subspace E of V, we have $\dim T(E) \leq \dim E$.) Therefore,

$$\operatorname{rank}(AB) \leq \dim A(\mathbb{R}^k) = \operatorname{rank}(A)$$

and

$$\operatorname{rank}(AB) \leq \dim B(\mathbb{R}^n) = \operatorname{rank}(B).$$

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