MATH 213, SPRING 2009
HOMEWORK 3 SOLUTIONS

1. (Chapter II, ex. 4.2) Recall that Taylor's theorem asserts that for a C^{k+1} function $h : \mathbb{R} \to \mathbb{R}$ and $\alpha \in \mathbb{R}$,
$$h(x) = \sum_{i=0}^{k} \frac{h^{(i)}}{i!}(x-\alpha)^i + \frac{1}{k!}\int_{\alpha}^{x} h^{(k+1)}(t)(x-t)^k \, dt.$$
We will use this result with $k = 1$.

   Let f be a C^∞ function defined in an open set $U \subset \mathbb{R}^n$. Let $a \in U$ be arbitrary and denote by B an open ball centered at a and contained in U. Fix $x \in B$ and define a function $\phi : [0, 1] \to \mathbb{R}$ by
$$\phi(t) = f(a + t(x-a)).$$
Note that ϕ is actually defined a small open neighborhood of the interval $[0, 1]$. Observe that
$$\phi'(t) = \sum_{i=1}^{n} \partial_i f(a + t(x-a))(x^i - a^i),$$
where $\partial_i f$ denotes the i^{th} partial derivative of f. It follows that
$$\phi''(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} \partial^2_{ji} f(a + t(x-a)) \; (x^i - a^i)(x^j - a^j),$$
where $\partial^2_{ji} f$ is the corresponding second order partial derivative of f. Applying Taylor's theorem of order two with $x = 1$ and $\alpha = 0$, we obtain
$$\begin{aligned}
f(x) &= \phi(1) \\
&= \phi(0) + \phi'(0) + \int_0^1 \phi''(t)(1-t)\, dt \\
&= f(a) + \sum_{i=1}^{n} \partial_i f(a)(x^i - a^i) + \int_0^1 \sum_{i=1}^{n}\sum_{j=1}^{n} \partial^2_{ji} f(a + t(x-a))(x^i - a^i)(x^j - a^j)(1-t)\, dt \\
&= f(a) + \sum_{i=1}^{n} \partial_i f(a)(x^i - a^i) + \frac{1}{2}\sum_{i,j=1}^{n} (x^i - a^i)(x^j - a^j) h^{ij}(x)
\end{aligned}$$
where
$$h^{ij}(x) = 2\int_0^1 \partial^2_{ji} f(a + t(x-a))(1-t)\, dt.$$
Since f is C^∞, so are h^{ij}. We also have
$$h^{ij}(a) = 2\int_0^1 \partial^2_{ij} f(a)(1-t)\, dt = \partial^2_{ij} f(a)\int_0^1 (1-t)\, dt = \partial^2_{ij} f(a),$$
as desired. □

2. (Chapter II, ex. 5.2) Recall that a vector field
$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x^i}$$
on an open set $U \subset \mathbb{R}^n$ is C^∞ iff each function $a_i : U \to \mathbb{R}$ is C^∞ on U.

($\Rightarrow$) Suppose Xf is C^∞, for each $f \in C^\infty(U)$. In particular, this holds for projections $p_i(x) = x^i$, $1 \leq i \leq n$ and $Xp_i = a_i$. Thus X is C^∞.

($\Leftarrow$) Suppose X is C^∞ on U. Then $X = \sum a_i \frac{\partial}{\partial x^i}$ and each a_i is C^∞ on U. It follows that

$$Xf = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x^i}.$$

All partial derivatives of a C^∞ function are by definition C^∞. Since products and sums of C^∞ functions are C^∞, it follows that $Xf \in C^\infty(U)$. □

3. (Chapter II, ex. 5.5) Let $A = C^\infty(\mathbb{R})$ be the algebra of C^∞ functions $\mathbb{R} \to \mathbb{R}$ and let $D = D_1 = D_2 = \partial/\partial x$ be the differentiation operator. Clearly, D is a derivation of A, but D^2 is not since it does not satisfy the Leibniz rule:

$$D^2(fg) = D^2 fg + 2DfDgfD^2 g \neq Dfg + fDg$$

unless $DfDg = 0$.

If D_1, D_2 are any two derivations of an arbitrary algebra, then $[D_1, D_2] = D_1D_2 - D_2D_1$ is clearly a linear map $A \to A$. It also satisfies the Leibniz rule: for any $a, b \in A$, we have

$$\begin{aligned}
[D_1D_2](ab) &= (D_1D_2 - D_2D_1)(ab) \\
&= D_1((D_2a)b + aD_2b) - D_2((D_1a)b + aD_1b) \\
&= ((D_1D_2a)b + D_2aD_1b + D_1aD_2b + aD_1D_2b) \\
&\quad - ((D_2D_1a)b + (D_1a)(D_2b) + (D_2a)(D_1b) + aD_2D_1b) \\
&= ([D_1, D_2]a)b + a([D_1, D_2]b).
\end{aligned}$$

Now let

$$X = \sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial x^i} \quad \text{and} \quad Y = \sum_{i=1}^{n} \beta^i \frac{\partial}{\partial x^i}$$

be two C^∞ vector fields on an open set $U \subset \mathbb{R}^n$. Then by the product rule, for any $f \in C^\infty(U)$,

$$\begin{aligned}
XYf &= X\left(\sum_{j=1}^{n} \beta^j \frac{\partial f}{\partial x^j}\right) \\
&= \sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial x^i}\left(\sum_{j=1}^{n} \beta^j \frac{\partial f}{\partial x^j}\right) \\
&= \sum_{i,j}\left(\alpha^i \frac{\partial \beta^j}{\partial x^i}\frac{\partial f}{\partial x^j} + \alpha^i\beta^j \frac{\partial^2 f}{\partial x^j \partial x^i}\right).
\end{aligned}$$

Similarly,

$$YXf = \sum_{i,j}\left(\beta^j \frac{\partial \alpha^i}{\partial x^j}\frac{\partial f}{\partial x^i} + \alpha^i\beta^j \frac{\partial^2 f}{\partial x^i \partial x^j}\right).$$

Using the equality of mixed partial derivatives (for this f need only be C^2), we obtain

$$(XY - YX)f = \sum_{i,j} \alpha^i \frac{\partial \beta^j}{\partial x^i}\frac{\partial f}{\partial x^j} - \sum_{i,j} \alpha^i \frac{\partial \beta^j}{\partial x^i}\frac{\partial f}{\partial x^j}.$$

Switching i and j in the first sum and factoring out $\partial f/\partial x^i$, we obtain

$$[X, Y]f = \sum_{i=1}^{n}\left\{\sum_{j=1}^{n}\left(\alpha^j \frac{\partial \beta^i}{\partial x^j} - \beta^j \frac{\partial \alpha^i}{\partial x^j}\right)\right\}\frac{\partial f}{\partial x^i},$$

where we denote $[X, Y] = XY - YX$. Therefore,

$$[X, Y] = \sum_{i=1}^{n} \gamma^i \frac{\partial}{\partial x^i},$$

where

$$\gamma^i = \sum_{j=1}^{n} \left(\alpha^j \frac{\partial \beta^i}{\partial x^j} - \beta^j \frac{\partial \alpha^i}{\partial x^j} \right) = X\beta^i - Y\alpha^i.$$

In particular, this means that $[X, Y]$ **is** a vector field on U, i.e., a derivation of $C^\infty(U)$. It is called the Lie bracket of X and Y. ☐

4. (Chapter II, ex. 6.4) If $K \subset \mathbb{R}^n$ is compact and $T : K \to K$ satisfies $d(Tx, Ty) < d(x, y)$ for all $x \neq y$ in K, then T has a unique fixed point.

   Proof. Define $\phi : K \to \mathbb{R}$ by $\phi(x) = d(Tx, x)$. Since T and $d : K \times K \to \mathbb{R}$ are continuous, so is ϕ. By compactness, ϕ attains its absolute minimum m at some point $x_0 \in K$. We claim that x_0 is a fixed point of T. Suppose not and set $y_0 = Tx_0$. Since $x_0 \neq y_0$, we have

$$\phi(y_0) = d(Ty_0, y_0) = d(Ty_0, Tx_0) < d(y_0, x_0) = d(Tx_0, x_0) = \phi(x_0),$$

   a contradiction. Therefore, $y_0 = x_0$, i.e., $Tx_0 = x_0$.

   Suppose x_1 is another fixed point of T. If $x_1 \neq x_0$, then

$$d(x_0, x_1) = d(Tx_0, Tx_1) < d(x_0, x_1),$$

   which is impossible. Therefore, $x_1 = x_0$ and T has a unique fixed point. ☐