MATH 213, SPRING 2009
HOMEWORK 4 SOLUTIONS

Chapter III, ex. 1.2: The straight line through N and $p \in U_N$ can be parametrized by $\ell(t) = N + t(p - N) = (0,0,1) + (x_1, x_2, x_3 - 1)$, where $p = (x_1, x_2, x_3)$. We are looking for the value of t such that the third coordinate $1 + t(x_3 - 1)$ of $\ell(t)$ is zero. This is clearly $t = 1/(1 - x_3)$, so

$$\varphi_N(x_1, x_2, x_3) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3} \right),$$

where we naturally identify the x_1x_2-plane $\mathbb{R}^2 \times \{0\}$ with $\mathbb{R}^2$. It is not hard to see that $\varphi_N : U_N \to \mathbb{R}^2$ is a bijection. To compute its inverse, observe that $\varphi_N^{-1}(u, v)$ has to be of the form

$$\varphi_N^{-1}(u, v) = s(u, v, 0) + (1 - s)(0, 0, 1) = (su, sv, 1 - s),$$

for some $s \in \mathbb{R}$. Furthermore, $\|(su, sv, 1 - s)\|^2 = s^2u^2 + s^2v^2(1 - s)^2 = 1$. Solving for s, we obtain

$$s = \frac{2}{1 + u^2 + v^2} = \frac{2}{1 + \|w\|^2},$$

where $w = (u, v)$. Therefore,

$$\varphi_N^{-1}(u, v) = \left(\frac{2u}{1 + \|w\|^2}, \frac{2v}{1 + \|w\|^2}, \frac{\|w\|^2 - 1}{1 + \|w\|^2} \right).$$

Since φ_N^{-1} is continuous, it follows that $\varphi_N : U_N \to \mathbb{R}^2$ is a homeomorphism.

In a completely analogous manner, we obtain

$$\varphi_S(x_1, x_2, x_3) = \left(\frac{x_1}{1 + x_3}, \frac{x_2}{1 + x_3} \right).$$

Let us compute the change of coordinates $\varphi_S \circ \varphi_N^{-1}$, defined on $\varphi_N(U_N \cap U_S) = \mathbb{R}^2 \setminus \{(0,0)\}$:

$$\begin{aligned}
\varphi_S \circ \varphi_N^{-1}(u, v) &= \varphi_S \left(\frac{2u}{1 + \|w\|^2}, \frac{2v}{1 + \|w\|^2}, \frac{\|w\|^2 - 1}{1 + \|w\|^2} \right) \\
&= \left(\frac{u}{\|w\|^2}, \frac{v}{\|w\|^2} \right) \\
&= \frac{w}{\|w\|^2},
\end{aligned}$$

which is clearly C^∞. Observe that $\varphi_S \circ \varphi_N^{-1}$ is the inversion relative to the unit circle in the x_1x_2-plane.

This construction easily generalizes to $S^n \subset \mathbb{R}^{n+1}$ if we take $N = (0, \ldots, 0, 1)$, $S = (0, \ldots, 0, -1)$ and define φ_N and φ_S analogously. Then

$$\varphi_N(x) = \left(\frac{x_1}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}} \right),$$

$$\varphi_S(x) = \left(\frac{x_1}{1 + x_{n+1}}, \ldots, \frac{x_n}{1 + x_{n+1}} \right),$$

and

$$\varphi_S \circ \varphi_N^{-1}(w) = \frac{w}{\|w\|^2},$$

which is just the inversion relative to the unit sphere in the plane $x_{n+1} = 0$. $\square$

Remark: Since $\|\varphi_N(p)\| \to \infty$, as $p \to N$, φ_N extends continuously to a homeomorphism of S^2 and the union of $\mathbb{R}^2$ and one point, usually denoted by ∞. Therefore, $\mathbb{R}^2$ can be *compactified* by one point; we need two points ($-\infty$ and $+\infty$) to compactify $\mathbb{R}$. The compactification is homeomorphic to S^2, which in this case is often called the Riemann sphere.

Chapter III, ex 1.3: Let $A \in \mathscr{M}^k_{m,n}(\mathbb{R})$ be arbitrary. Since the rank of A is $\geq k$, there exists at least one $k \times k$ submatrix M of A such that $\det M \neq 0$. Assume that M is the intersection of rows $(m_1, \dots, m_k)$ and columns $(n_1, \dots, n_k)$ of A. Since $X \mapsto \det X$ is a continuous function, there exists a neighborhood $\mathscr{U}$ of M in $\mathscr{M}_k(\mathbb{R})$ such that $\det N \neq 0$, for all $N \in \mathscr{U}$. Let $\mathscr{V}$ be the open subset of $\mathscr{M}_{m,n}(\mathbb{R})$ consisting of all matrices whose corresponding $(m_1, \dots, m_k) \times (n_1, \dots, n_k)$ submatrix (in the above sense) lies in $\mathscr{U}$. Then for all $B \in \mathscr{V}$, the rank of B is $\geq k$, hence $B \in \mathscr{M}^k_{m,n}(\mathbb{R})$. This implies that $\mathscr{V} \subset \mathscr{M}^k_{m,n}(\mathbb{R})$, proving that $\mathscr{M}^k_{m,n}(\mathbb{R})$ is an open submanifold of $\mathscr{M}_{m,n}(\mathbb{R})$.

Alternatively, one could show that the complement $\mathscr{M}^k_{m,n}(\mathbb{R})^c$ of $\mathscr{M}^k_{m,n}(\mathbb{R})$ (which is the set of all $m \times n$ matrices whose rank is $< k$) is closed in the following way. Define a function $f : \mathscr{M}_{m,n}(\mathbb{R}) \to \mathbb{R}$ by

$$f(A) = \sum_M (\det M)^2,$$

where the summation is over all $k \times k$ submatrices (i.e., minors) M of A. Since f is a polynomial function of the entries of A, it is continuous. Observe that $f(A) = 0$ iff the rank of A is $< k$. Therefore, $f^{-1}(\{0\}) = \mathscr{M}^k_{m,n}(\mathbb{R})^c$ is a closed set. $\square$

Chapter III, ex. 2.3: The relation is clearly reflexive and symmetric. Let us show that it is transitive. Suppose $x \sim y$ and $y \sim z$. If either $x = y$ or $y = z$, then $x \sim z$ trivially. So suppose that $y = \alpha(x) = -x$ and $z = \alpha(y) = -y$. Then $z = x$, so clearly $x \sim z$.

Denote the equivalence classes of $\sim$ by $\{x\}$ and denote the equivalence classes in $P^n(\mathbb{R})$ by $[x]$; this means that $[x]$ is the unique straight line in $\mathbb{R}^{n+1}$ passing through x and the origin.

We can identify $S^n/\sim$ with $P^n(\mathbb{R})$ via a map $f : S^n/\sim \longrightarrow P^n(\mathbb{R})$ defined by

$$f(\{x\}) = [x].$$

Since both x and $-x$ define the same line, f is well-defined. It is 1–1, because $f(\{x\}) = f(\{y\})$ implies $[x] = [y]$, so $y = \pm x$, hence $x \sim y$ and $\{x\} = \{y\}$. It is clearly onto and continuous. Recall that a 1–1 continuous map from a compact space onto a Hausdorff space is automatically a homeomorphism. Thus, f is a homeomorphism. $\square$

Chapter III, ex. 2.6: Since F is constant on equivalence classes and $F'([x]) = F(x)$, F' is well-defined.

Let us show that F' is continuous. Suppose that $V \subset Y$ is open. Set $U = (F')^{-1}(V)$. Then U is open in $X/\sim$ iff $\pi^{-1}(U)$ is open in X. But

$$\pi^{-1}(U) = F^{-1}(V),$$

which is open in X by continuity of F. $\square$