MATH 213, SPRING 2009
HOMEWORK 5 SOLUTIONS

Chapter III, ex. 3.1: ($\Rightarrow$) Suppose $F : M \to N$ is C^∞. Then $f \circ F$ is C^∞, since the composition of C^∞ maps is C^∞, for any C^∞ $f : W \to \mathbb{R}$, where $W \subset N$ is open.

($\Leftarrow$) Suppose $f \circ F$ is C^∞ for every such $f : W \to \mathbb{R}$. Let (U, φ) and (V, ψ) be arbitrary charts for M and N, respectively. Denote the components of ψ by $\psi^1, \dots, \psi^n$, so that $\psi(q) = (\psi^1(q), \dots, \psi^n(q))$, for every $q \in V$. Our assumption implies that $\psi^i \circ F$ is C^∞, for every $1 \leq i \leq n$. This by definition means that $\psi^i \circ F \circ \varphi^{-1}$ is C^∞, for each i, which in turn implies that

$$\psi \circ F \circ \varphi^{-1} = (\psi^1 \circ F \circ \varphi^{-1}, \dots, \psi^n \circ F \circ \varphi^{-1})$$

is C^∞. Since the expression of F in any pair of coordinate neighborhoods is C^∞, it follows that F is itself C^∞, by definition. □

Chapter III, ex. 3.7: To show that $\pi : F(k, n) \to G(k, n)$ is C^∞, we need to express it in local coordinates. Recall that they are defined as follows. For any ordered k-tuple $(j_1, \dots, j_k)$ of elements of $\{1, \dots, n\}$, with $j_1 < \cdots < j_k$, we define $\tilde{U}_J \subset F(k, n)$ as the set of all frames (thought of as matrices) $x \in F(k, n)$ whose submatrix corresponding to J is non-singular. Let $\tilde{\varphi}_J : \tilde{U}_J \to \mathbb{R}^{kn}$ be the usual identification of $k \times n$ matrices with kn-vectors. We then define

$$U_J = \pi(\tilde{U}_J) \qquad \text{and} \qquad \varphi_J([x]) = p_J(x_J^{-1} x).$$

where $p_J : \tilde{U}_J \to \mathbb{R}^{k(n-k)}$ is the map that takes the $k \times n$ matrix $x_J^{-1} x$ to the submatrix corresponding to the complement of J, followed by the usual identification map of $k \times (n-k)$ matrices with vectors in $\mathbb{R}^{k(n-k)}$. The expression of π relative to $(\tilde{U}_J, \tilde{\varphi}_J)$ and (U_J, φ_J) is

$$\begin{aligned} \hat{\pi}(v) &= \varphi_J \circ \pi \circ \tilde{\varphi}_J^{-1}(v) \\ &= \varphi_J([x]) \\ &= p_J(x_J^{-1} x), \end{aligned}$$

where $\tilde{\varphi}_J(x) = v \in \mathbb{R}^{kn}$. To complete the proof we need the following

Lemma. *Each entry of the matrix $x_J^{-1} x$ is a C^∞ function of the entries of the matrix $x \in \tilde{U}_J$.*

Proof of the lemma. By linear algebra, each entry of $x_J^{-1} x$ is a rational function of the entries of x. Rational functions are C^∞ (on their domain). □

Chapter III, ex. 4.3: Denote by F the 1–1 immersion $\mathbb{R} \to \mathbb{R}^2$ whose image is the figure eight in Example 4.9. Let $A = F(-1, 1)$. Then A is an arc (contained in the second and fourth quadrants and centered at the origin). Since the topology of $\tilde{N}$ is defined via F, A is connected since $F^{-1}(A)$ is. Consider, $B = H(A)$, which is also an arc centered at the origin but contained in the first and third quadrant. It is not hard to see that $F^{-1}(B)$ is a union of the type $(-\infty, -a) \cup (b, \infty)$, for some $a, b > 0$. Thus B is *disconnected*. Since H maps a connected set to a disconnected one, it is not continuous, hence not a homeomorphism (not to mention a diffeomorphism) of $\tilde{N}$.

Chapter III, ex. 4.7: We need to express F is local coordinates. Recall that S^{n-1} is covered by coordinate neighborhoods $(U_i^\pm, \varphi_i^\pm)$, where $U_i^\pm = \{x \in S^{n-1} : \pm x_i > 0\}$ and $\varphi_i^\pm : U_i^\pm \to \mathbb{R}^{n-1}$ is the corresponding projection along the x_i-axis to the unit disk in $\mathbb{R}^{n-1}$. Recall also that the standard atlas of $P^{n-1}(\mathbb{R})$ consists of coordinate neighborhoods (V_i, ψ_i), where $V_i = \{[x] \in P^{n-1}(\mathbb{R}) : x_i \neq 0\}$ and $\psi_i : V_i \to \mathbb{R}^{n-1}$ is defined by $\psi_i([x_1, \ldots, x_n]) = (x_1/x_i, \ldots, x_{i-1}/x_i, \hat{1}, x_{i+1}/x_i, \ldots, x_n/x_i)$.

Observe that $F(U_i^\pm) \subset V_i$ and consider $\hat{F} = \psi_n \circ F \circ (\varphi_n^+)^{-1} : \varphi_i^+(U_n^+) \to \psi_n(V_n)$:

$$\begin{aligned}\hat{F}(u_1, \ldots, u_{n-1}) &= (\psi_n \circ F)(u_1, \ldots, u_{n-1}, \sqrt{1 - u_1^2 - \cdots - u_{n-1}^2}) \\ &= \psi_n([u_1, \ldots, u_{n-1}, \sqrt{1 - u_1^2 - \cdots - u_{n-1}^2}]) \\ &= \frac{1}{\sqrt{1 - u_1^2 - \cdots - u_{n-1}^2}}(u_1, \ldots, u_{n-1}).\end{aligned}$$

That is,

$$\hat{F}(u) = \frac{u}{\sqrt{1 - \|u\|^2}}.$$

It is not hard to see that this map of the open unit disk D^{n-1} is C^∞, 1–1 and onto $\mathbb{R}^{n-1}$. Its inverse is

$$v \mapsto \frac{v}{\sqrt{1 + \|v\|^2}},$$

which is also C^∞. Therefore, $\hat{F}$ is a diffeomorphism from D^{n-1} to $\mathbb{R}^{n-1}$, hence has rank $n-1$.

In a completely analogous fashion it can be shown that the rank of the expression of F in all other pairs of coordinate neighborhoods has rank $n-1$. This, by definition, implies that the rank of F is everywhere $n-1$.

Remark: This is always the case for quotient maps of manifolds. Namely, let M be smooth manifold with an equivalence relation $\sim$ such that $M/\sim$ is a smooth manifold. Then the natural projection $\pi : M \to M/\sim$ is a submersion.