MATH 213, SPRING 2009
HOMEWORK 6 SOLUTIONS

Chapter III, ex. 4.6: Suppose $f : M \to N$ and $g : N \to L$ are immersions and let $h = g \circ f$. This means that the rank of f is $m = \dim M$ and the rank of g is $n = \dim N$, everywhere. Let $p \in M$ be arbitrary and let (U, ϕ) and (V, ψ) be arbitrary charts containing p and $q = h(p)$, respectively, such that $h(U) \subset V$. Then $\hat{h}$ is the local representation of h in these charts. We claim that the rank of $\hat{h}$ is m at every point.

Let (W, χ) be a chart around $w = f(p)$ (in N). Assume $f(U) \subset W$ and $g(W) \subset V$. Denote the local representations of f and g in their corresponding charts by $\hat{f}$ and $\hat{g}$. Then

$$\hat{h} = \hat{g} \circ \hat{f}.$$

Differentiating, we obtain

$$D\hat{h}(x) = D\hat{g}(f(x))D\hat{f}(x)$$

By assumption, the rank of $D\hat{g}$ is n and the rank of $D\hat{f}$ is m, everywhere.

Lemma. *A linear map $L : V \to W$ between finite dimensional vector spaces is 1–1 iff* $\operatorname{rank} L = \dim V$.

Proof. Recall that $L(V)$ is isomorphic to $V/\operatorname{Ker}(L)$, hence $\dim L(V) = \dim V - \dim \operatorname{Ker}(L)$, where $\operatorname{Ker}(L)$ is the kernel (nullspace) of L.

($\Rightarrow$) Suppose L is 1–1. Then L has a trivial kernel, so $\dim V = \dim L(V) = \operatorname{rank} L$.

($\Leftarrow$) Suppose the rank of L equals $\dim V$. Then $\dim L(V) = \dim V$. This implies that $\operatorname{Ker}(L)$ (the kernel of L) is trivial. Thus L is 1–1. □

The Lemma implies that $D\hat{g}(y)$ and $D\hat{f}(x)$ are 1–1, for all x, y. Since the composition of 1–1 maps is 1–1, it follows that $D\hat{h}(x)$ is 1–1, hence has rank m (again by the Lemma). Therefore, h is an immersion. □

Chapter III, ex. 5.1: ($\Rightarrow$) Suppose that $F : N \to M$ is a proper 1–1 immersion. To show that F is an embedding, it suffices to show that F is a closed map, i.e., that $F(A)$ is closed in M, for every closed set A in N. Suppose not, i.e., assume that there exists a closed set $A \subset N$ such that $B = F(A)$ is not closed in M. Then B does not contain at least one of its accumulation (or cluster) points, call it y. That is, there exists a sequence (y_k) in B, which converges to y in M, but $y \notin B$. Since F is 1–1 and $y_k \in B = F(A)$, there exists a (unique) sequence (x_k) in A such that $y_k = F(x_k)$. Let

$$K = \{y, y_1, y_2, y_3, \ldots\}.$$

Since $y_k \to y$, K is compact (every sequence in it has a convergent subsequence). The assumption that F is proper implies that $F^{-1}(K)$ is compact. Since (x_k) is a sequence in $F^{-1}(K)$, it has a convergent subsequence. We denote it by (x_{k_i}) and call its limit x. Observe that $x_{k_i} \in A$ for every i, so x is an accumulation point of A, hence in A, since A is closed. On the other hand, by continuity of F,

$$F(x) = \lim_{i \to \infty} F(x_{k_i}) = \lim_{i \to \infty} y_{k_i} = y.$$

But this is a contradiction, since $F(x) \in B$ and we assumed that $y \notin B$! Therefore, F is a closed map, hence an embedding.

The remaining the conclusions follow readily: (i) since F is an embedding, $F(N)$ is an embedded, hence regular submanifold of M; (ii) it is closed as the image of N, which is closed in itself.

($\Leftarrow$) If F is an embedding and $\tilde{N}$ is a closed, regular submanifold of M, then for every compact set $K \subset M$, we have

$$F^{-1}(K) = F^{-1}(K \cap \tilde{N}),$$

where the first F^{-1} denotes the preimage by F and the second one denotes the *inverse* of F. Since $F^{-1} : \tilde{N} \to N$ is continuous and $K \cap \tilde{N}$ is compact (as a closed subset of the compact set K – closedness of $\tilde{N}$ is used here), it follows that $F^{-1}(K)$ is compact. This proves that F is proper. □

Chapter III, ex. 5.6: Let $M = \mathbb{R}^2$ and let N be the figure eight from example III.4.9. Recall that N is the image of a 1–1 immersion $G : \mathbb{R} \to \mathbb{R}^2$, where $G(0) = (0, 0)$. Define a function $f : N \to \mathbb{R}$ by

$$f(G(t)) = t,$$

that is, $f = G^{-1}$. Then f is smooth as a function on N since $f \circ G^{-1} = \text{identity} : \mathbb{R} \to \mathbb{R}$ is smooth.

We claim that f is not the restriction of a smooth function $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}$. Suppose it is, i.e., there exists a smooth function $\tilde{f}$ on $\mathbb{R}^2$ such that $\tilde{f}(p) = f(p)$, for all $p \in N$. On one hand,

$$\tilde{f}(0, 0) = f(0, 0) = f(G(0)) = 0.$$

Now let $p_n = G(n)$. As $n \to \infty$, $p_n \to (0, 0)$. Since $\tilde{f}$ is continuous (being smooth), we have:

$$\lim_{n\to\infty} \tilde{f}(p_n) = \tilde{f}(0, 0) = 0.$$

But $\tilde{f}$ agrees with f on the sequence (p_n) (which lies in N), so $\tilde{f}(p_n) = f(p_n) = f(G(n)) = n \to \infty$. Contradiction! Therefore, there is no smooth (even continuous!) function $\tilde{f}$ on $\mathbb{R}^2$ whose restriction to N equals f. □

Chapter III, ex. 5.7: Since $P^2(\mathbb{R})$ is compact, by Theorem 5.7 it suffices to show that $\tilde{F}$ is a 1–1 immersion. Suppose $\tilde{F}([x]) = \tilde{F}([y])$, for some $x, y \in \mathbb{R}^3$. Then $F(x) = F(y)$, so

$$x_1^2 - x_2^2 = y_1^2 - y_2^2, \quad x_1x_2 = y_1y_2, \quad x_1x_3 = y_1y_3, \quad x_2x_3 = y_2y_3.$$

At least one of the y_i's is nonzero. Suppose, for instance, that $y_1 \neq 0$, other cases are handled completely analogously. Then

$$y_2 = \frac{x_1}{y_1}x_2 \quad \text{and} \quad y_3 = \frac{x_1}{y_1}x_3.$$

Substituting into the first equation and rearranging terms, we obtain

$$(x_2^2 + y_1^2)(x_1^2 - y_1^2) = 0.$$

If $x_1^2 - y_1^2 \neq 0$, then $x_2 = y_1 = 0$, contrary to our assumption. Therefore, $y_1 = \pm x_1$, so $y_2 = \pm x_2$ and $y_3 = \pm x_3$. This implies $x = \pm y$, so $[x] = [y]$. Thus $\tilde{F}$ is 1–1.

To find the rank of $\tilde{F}$, we need to express it in local coordinates. Let (V_i, ψ_i) be coordinate neighborhoods as in exercise 4.7, with $n = 3$. Then

$$\begin{aligned}
\hat{F}(u_1, u_2) &= (\tilde{F} \circ \psi_3^{-1})(u_1, u_2) \\
&= \tilde{F}([u_1, u_2, 1]) \\
&= F(u_1, u_2, 1) \\
&= (u_1^2 - u_2^2, u_1u_2, u_1, u_2).
\end{aligned}$$

Hence

$$D\hat{F}(u_1, u_2) = \begin{bmatrix} 2u_1 & -2u_2 \\ u_2 & u_1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which clearly has rank 2. It can be shown similarly that $\tilde{F} \circ \psi_i$ has the same rank for all i. Therefore, $\tilde{F}$ is an immersion, hence, being 1–1, an embedding.