MATH 213, SPRING 2009
HOMEWORK 7 SOLUTIONS

Chapter III, ex. 6.3: Since left translations $L_a : G \to G$ are diffeomorphisms, $L_a(U)$ is open for every open set U in G. Therefore,

$$AU = \bigcup_{a \in A} L_a(U)$$

is open, as a union of open sets. □

Chapter III, ex. 7.4: To show that the action of $O(n)$ on S^{n-1} is transitive, it is enough to show that the orbit of $e_1 = (1, 0, \dots, 0)$ is all of S^{n-1}. Let $v \in S^{n-1}$ be arbitrary. By linear algebra, we can complete $v_1 = v$ to an orthonormal basis $\{v_1, v_2, \dots, v_n\}$ of $\mathbb{R}^n$. Let A be the matrix with columns $v_1, \dots, v_n$. By construction, $A \in O(n)$ and $Ae_1 = v_1$. Therefore, v_1 is in the $O(n)$-orbit of e_1.

Suppose $T \in O(n)$ fixes e_1, i.e., $Te_1 = e_1$. This implies that the $(n-1)$-dimensional plane $e_1^\perp$ orthogonal to e_1 is also invariant under T, $Te_1^\perp = e_1^\perp$, so T is of the form

$$T = \begin{bmatrix} 1 & \mathbf{0}_{1 \times (n-1)} \\ \mathbf{0}_{(n-1) \times 1} & S \end{bmatrix},$$

where $S \in O(n-1)$. Therefore, the isotropy subgroup of e_1 is isomorphic to $O(n-1)$. □

Chapter III, ex. 7.5: The action of $Gl(n, \mathbb{R})$ on $P^{n-1}(\mathbb{R})$ is given by

$$A[x] = [Ax],$$

where $[x]$ denotes the equivalence class of $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ in $P^{n-1}(\mathbb{R})$. To show this action is transitive, it suffices to show that the orbit of $[e_1]$ is the whole projective space, where $e_1 = (1, 0, \dots, 0)$. So let $[x] \in P^{n-1}(\mathbb{R})$ be arbitrary. By linear algebra, we can complete $x_1 = x$ to a basis $\{x_1, x_2, \dots, x_n\}$ of $\mathbb{R}^n$. Let A be the matrix with columns $x_1, \dots, x_n$. By construction, $A \in Gl(n, \mathbb{R})$ and $Ae_1 = x$, hence $A[e_1] = [x]$. Therefore, $[x]$ is in the orbit of $[e_1]$.

Now assume T is in the isotropy group of $[e_1]$, i.e., $T[e_1] = [e_1]$, for some $T \in Gl(n, \mathbb{R})$. This means that $Te_1 = \lambda e_1$, for some real number $\lambda \neq 0$, i.e., e_1 is an eigenvector of T. Therefore, the $(1, 1)$-entry of any T in $Gl(n, \mathbb{R})_{[e_1]}$ is nonzero and all other entries in the first column are zero. This defines a closed subgroup of $Gl(n, \mathbb{R})$.

Remark: Denote the eigenvalue of T corresponding to e_1 by $\lambda = \lambda(T)$. It is easy to see that $\lambda : G \to \mathbb{R}^*$ is a homomorphism of groups, where $G = Gl(n, \mathbb{R})_{[e_1]}$ is the isotropy subgroup of $[e_1]$ and $\mathbb{R}^*$ is the multiplicative group of non-zero real numbers.

Let E_1 be the line spanned by e_1 and let $V = \mathbb{R}^n / E_1$ be the quotient vector space consisting of equivalence classes $[v] = v + E_1$. Denote by $Gl(V)$ the (Lie) group of linear automorphisms of V; observe that $Gl(V)$ is isomorphic to $Gl(n-1, \mathbb{R})$ (it becomes $Gl(n-1, \mathbb{R})$ once we choose a basis of V). Define a map $\pi : G \to Gl(V)$ by $\pi(T) = [T]$, where

$$[T][v] = [Tv].$$

It is not hard to check that $[T]$ is well-defined and a linear isomorphism of V. Furthermore, it is straightforward to check that π is a group homomorphism. Now define

$$\phi : G \to \mathbb{R}^* \times Gl(V)$$

by $\phi(T) = (\lambda(T), \pi(T))$. It can be easily verified that ϕ is a group isomorphism. Therefore, the isotropy group of $[e_1]$ is isomorphic to $\mathbb{R}^* \times Gl(n-1, \mathbb{R})$. $\square$

Chapter III, ex. 7.8: Suppose that $H < G$ acts on G by left translations and assume $hg = g$, for some $h \in H$ and $g \in G$ (that is, g is a fixed point for h). Then, clearly, $h = e$, showing that the action of H on G is *free*.

Suppose now that G acts freely and transitively on the left on X, $(g, x) \mapsto gx$. Let $x_0 \in X$ be a point with whose G-orbit is all of X. Define a map $\phi : G \to X$ by

$$\phi(g) = gx_0.$$

We claim that ϕ is a bijection. Suppose $\phi(g_1) = \phi(g_2)$. Then $g_1x_0 = g_2x_0$, so $g_2^{-1}g_1x_0 = x_0$. Since the action is free, $g_2^{-1}g_1 = e$, hence $g_1 = g_2$. Thus ϕ is injective. Surjectiveness follows from transitivity of the action: for an arbitrary $x \in X$ there exists $g \in G$ such that $gx_0 = x$, i.e., $\phi(g) = x$.

If we identify X with G via the map ϕ, that is, $\phi(g) \equiv g$, then it is obvious that the given action is equivalent to the action of G on itself by left translations. $\square$