MATH 213, SPRING 2009
HOMEWORK 8 SOLUTIONS

Chapter IV, ex. 2.5: We topologize TM by the declaring that each map $\tilde{\varphi} : \tilde{U} \to U \times \mathbb{R}^n$ be a homeomorphism.

To show that the collection $\tilde{\mathscr{A}} = \{(\tilde{U}, \tilde{\varphi})\}$ is an atlas for TM, it suffices to prove that the coordinate changes are C^∞, since $\tilde{\mathscr{A}}$ clearly covers TM. Let (U, φ) be a chart for M and let $\{E_1, \dots, E_n\}$ be the coordinate frame defined by φ on U. Let

$$v = \sum_{i=1}^{n} \alpha_i E_i \in T_pM.$$

Observe that $\tilde{\varphi}(v) = (\varphi(p), \alpha_1, \dots, \alpha_n)$ considered as a vector in $T_{\varphi(p)}\mathbb{R}^n$ can be written as

$$\sum_{i=1}^{n} \alpha_i \left.\frac{\partial}{\partial x_i}\right|_{\varphi(p)}.$$

In other words,

$$\tilde{\varphi}\left(\sum_{i=1}^{n} \alpha_i E_{ip}\right) = \sum_{i=1}^{n} \alpha_i \left.\frac{\partial}{\partial x_i}\right|_{\varphi(p)} = \varphi_*\left(\sum_{i=1}^{n} \alpha_i E_{ip}\right).$$

This implies that $\tilde{\varphi} = \varphi_*$, the derivative of φ. Therefore, for any chart (V, ψ) of M, the coordinate changes in TM are just

$$\tilde{\psi} \circ \tilde{\varphi}^{-1} = \psi_* \circ \varphi_*^{-1} = (\psi \circ \varphi^{-1})_* = D(\psi \circ \varphi^{-1}),$$

hence clearly C^∞ since $\psi \circ \varphi^{-1}$ is C^∞. □

Chapter IV, ex. 2.6: To show that $\pi : TM \to M$ is a submersion, we have to show that the rank of π is maximal, i.e., equal to $n = \dim M$. Let (U, φ) be a coordinate neighborhood of an arbitrary point $p \in M$ and let $(\tilde{U}, \tilde{\varphi})$ be the corresponding coordinate neighborhood for TM. Then for every $(x, \alpha) \in U \times \mathbb{R}^n$, with $\alpha = (\alpha_1, \dots, \alpha_n)$, we have

$$\begin{aligned}\hat{\pi}(x, \alpha) &= (\varphi \circ \pi \circ \tilde{\varphi}^{-1})(x, \alpha) \\ &= (\varphi \circ \pi)\left(\sum \alpha_i E_{i,\varphi^{-1}(x)}\right) \\ &= \varphi(\varphi^{-1}(x)) \\ &= x,\end{aligned}$$

i.e., $\tilde{\pi}$ is the projection $U \times \mathbb{R}^n \to U$ to the first component, which clearly has rank n. By Corollary III.5.9, $\pi^{-1}(p) = T_pM$ is a regular submanifold of TM of dimension $\dim TM - \dim M = \dim M = n$. □

Chapter IV, ex. 3.5: Since

$$\theta_t(x, y) = (x \cos t + y \sin t, -x \sin t + y \cos t)^T = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

the action is global as $\theta_s \circ \theta_t = \theta_{s+t}$ is clearly true for all $s, t \in \mathbb{R}$.

The infinitesimal generator of θ is

$$X(x,y) = \left.\frac{d}{dt}\right|_0 \theta_t(x,y) = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \equiv (y,-x)^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Since θ_t is the rotation by t radians around the origin, each orbit of θ is a circle centered at the origin of radius $r \geq 0$. □

Chapter IV, ex. 3.7: The infinitesimal generator of the given action is

$$\begin{aligned} \left.\frac{d}{dt}\right|_0 \theta(t,A) &= \left.\frac{d}{dt}\right|_0 \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} A \\ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} A \\ &\equiv a_{21}\frac{\partial}{\partial x_{11}} + a_{22}\frac{\partial}{\partial x_{12}}, \end{aligned}$$

where $A = [a_{ij}]$. Observe that $\theta(t,A) = \exp\left(t\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) A.$ □