MATH 213, SPRING 2009
HOMEWORK 9 SOLUTIONS

Chapter IV, ex. 4.4: ($\Rightarrow$) Suppose $F_*(X) = Y$. Let $p \in M$ be arbitrary but fixed and set $q = F(p)$. Define a curve α on N by

$$\alpha(t) = F(\theta_t(p)).$$

Then $\alpha(0) = F(p) = q$ and since $\frac{d}{dt}\theta_t(p) = X(\theta_t(p))$, we have

$$\begin{aligned}\dot{\alpha}(t) &= F_*(X(\theta_t(p))) \\ &= Y(F(\theta_t(p))) \\ &= Y(\alpha(t)).\end{aligned}$$

This means that α is an integral curve of Y passing through q at time $t = 0$. But $t \mapsto \sigma_t(q)$ is another such integral curve. By uniqueness of integral curves of smooth vector fields, we have $\alpha(t) = \sigma_t(q)$, for all t for which both sides are defined (if M and N are compact, then this holds for all $t \in \mathbb{R}$). This proves the first part.

($\Leftarrow$) Suppose $F(\theta_t(p)) = \sigma_t(F(p))$, for all $(t, p) \in \mathbb{R} \times M$ for which both sides are defined. Then differentiation with respect to t at zero yields

$$\begin{aligned}Y(F(p)) &= \left.\frac{d}{dt}\right|_0 \sigma_t(F(p)) \\ &= \left.\frac{d}{dt}\right|_0 F(\theta_t(p)) \\ &= F_*(\left.\frac{d}{dt}\right|_0 \theta_t(p)) \\ &= F_*(X(p)).\end{aligned}$$

Thus $F_*(X) = Y$. □

Chapter IV, ex. 6.4: It is easy to see that $A^2 = -I$, $A^3 = -A$, and $A^4 = I$. It follows that

$$A^{2k} = (-1)^k I \quad \text{and} \quad A^{2k+1} = (-1)^k A,$$

for all $k \in \mathbb{N}$. Therefore,

$$\begin{aligned}e^{tA} &= \sum_{n=0}^{\infty} \frac{(tA)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} A^{2k} + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} A^{2k+1} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} I + \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k=1}}{(2k+1)!} A \\ &= (\cos t) I + (\sin t) A \\ &= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.\end{aligned}$$

This is the matrix corresponding to the rotation by the angle of t radians. The action of the 1-parameter subgroup defined by A is given by $\phi_t(p) = e^{tA}p$. Its orbits are circles and the origin (the only fixed point of the action). The infinitesimal generator of the action is by definition A, i.e., the linear vector field $X(x,y) = A(x,y)^T = (y,-x)^T$, or equivalently

$$X(x,y) = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

It is easy to see that $B^2 = 0$, so

$$e^{tB} = I + tB = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

The action of the 1-parameter subgroup of $Gl(2,\mathbb{R})$ defined by B is given by

$$\psi_t(x,y) = e^{tB}\begin{bmatrix} x \\ y \end{bmatrix} = (x+ty, y)^T.$$

The orbits are horizontal lines $y = \text{constant} \neq 0$ and fixed points $(x,0)$, for all $x \in \mathbb{R}$. The infinitesimal generator of the action is B, i.e., the linear vector field $Y(x,y) = B(x,y)^T = (y,0)$, or equivalently,

$$Y(x,y) = y\frac{\partial}{\partial x}.$$

Chapter IV, ex. 6.6: Suppose A is a non-singular $n \times n$ matrix and $X \in \mathscr{M}_n(\mathbb{R})$. It is easy to see that $(AXA^{-1})^k = AX^kA^{-1}$, for all $k \in \mathbb{N}$, so

$$\begin{aligned} e^{AXA^{-1}} &= \sum_{k=0}^{\infty} \frac{(AXA^{-1})^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{AX^kA^{-1}}{k!} \\ &= A\left\{\sum_{k=0}^{\infty} \frac{X^k}{k!}\right\} A^{-1} \\ &= Ae^XA^{-1}. \end{aligned}$$

Observe that this also holds for complex matrices A.

Let $X \in \mathscr{M}_n(\mathbb{R})$ be arbitrary. There exists a nonsingular complex matrix A such that $T = AXA^{-1}$ is an upper triangular matrix with the eigenvalues $\lambda_1, \dots, \lambda_n$ of X on the main diagonal. It is easy to see that e^T is upper triangular as well, with $e^{\lambda_1}, \dots, e^{\lambda_n}$ on the main diagonal. Recall that the determinant of a triangular matrix is the product of the entries on its main diagonal. It follows that

$$\begin{aligned} \det e^X &= \det(Ae^XA^{-1}) \\ &= \det e^{AXA^{-1}} \\ &= \det \operatorname{diag}(e^{\lambda_1}, \dots, e^{\lambda_n}) \\ &= e^{\lambda_1} \cdots e^{\lambda_n} \\ &= \exp(\lambda_1 + \dots + \lambda_n) \\ &= \exp(\operatorname{trace}(X)). \end{aligned}$$

Suppose now that $t \mapsto e^{tA}$ is a 1-parameter subgroup of $Sl(n,\mathbb{R})$. Then $\det e^{tA} = 1$, for all t. The just proved identity implies that $\operatorname{trace}(tA) = 0$, for all t, which means that A has trace zero. It follows that the 1-parameter subgroups of $Sl(n,\mathbb{R})$ are in 1–1 correspondence with the space of traceless $n \times n$ matrices. $\square$

Chapter IV, ex. 7.2: Recall that $[X,Y] = L_X Y$ and

$$[X,Y](p) = \frac{d}{dt}\bigg|_0 (\phi_{-t})_*(Y(\phi_t(p))),$$

where $\{\phi_t\}$ is the (local) flow of X.

Assume now that X, Y are smooth vector fields and g is a smooth function on M. Let $\{\phi_t\}$ is the (local) flow of X. Then:

$$\begin{aligned}
[X, gY](p) &= \frac{d}{dt}\bigg|_0 (\phi_{-t})_*[g(\phi_t(p))Y(\phi_t(p))] \\
&= \frac{d}{dt}\bigg|_0 g(\phi_t(p))(\phi_{-t})_*(Y(\phi_t(p))) \\
&= \frac{d}{dt}\bigg|_0 g(\phi_t(p))Y(p) + g(p)\,\frac{d}{dt}\bigg|_0 (\phi_{-t})_*(Y(\phi_t(p))) \\
&= (Xg)(p)Y(p) + g(p)(L_X Y)(p) \\
&= (Xg)(p)Y(p) + g(p)[X,Y](p).
\end{aligned}$$

If f is a smooth function on M, then by antisymmetry of the Lie bracket,

$$[fX, Y] = -[Y, fX] = -(Yf)X - f[X,Y].$$

Combining these two results and setting $\tilde{X} = fX$, we obtain:

$$\begin{aligned}
[fX, gY] &= [\tilde{X}, gY] \\
&= (\tilde{X}g)Y + g[\tilde{X}, Y] \\
&= f(Xg)Y + g[fX, Y] \\
&= f(Xg)Y - g(Yf)X - gf[X,Y].
\end{aligned}$$

Here we used the fact that $(fX)g = f(Xg)$. This proves the desired identity. $\square$