Operations on Manifolds and Minimal Presentations [1]

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1 Introduction

First we describe various ways of gluing manifolds together: connected sum, connected sum along the boundary, attachment of handles, etc. A brief discussion of the effect of these operations on homology prepares the ground for the more precise results to follow. Then we describe a way to build some highly connected manifolds; it turns out later that this method is, in a sense, generic. That is, all highly connected manifolds can be constructed in this way.

Along the way, an important result describes the situation when two successive attachments of handles produces no change: The second handle destroys the first. This is Smale's Cancellation Lemma.

The main idea here, concerns the existence of the handle presentation with the minimal number of handles determined by its homology groups. The following example should explain the importance of this idea. The minimal number of handles necessary to build an $n-$dimensional sphere is two: two $n-$discs glued along boundaries. If we suceed in proving that a homotopy sphere admits a presentation with the minimal number of handles determined by its homology, then it must admit a presentation with two handles. In turn, this implies that it is homeomorphic to the sphere, i.e., the Poincare conjecture.

2 Operations on manifolds

Given two connected $m-$dimensional manifolds M_1, M_2, let $h_i : R^m \to M_i, i = 1, 2$, be two imbeddings. Let $\alpha : (0, \infty) \to (0, \infty)$ be an arbitrary orientation reversing diffeomorphism. We define $\alpha_m : \mathbf{R}^m - \mathbf{0} \to \mathbf{R}^m - \mathbf{0}$ by

$$\alpha_m(v) = \alpha(|v|)\frac{v}{|v|} \tag{2.1}$$

The connected sum $M_1 \# M_2$ is the space obtained from the (disoint) union of $M_1 - h_1(\mathbf{0})$ and $M_2 - h_2(\mathbf{0})$ by identifying $h_1(v)$ with $h_2(\alpha_m(v))$. It turns out that $M_1 \# M_2$

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does not depend on the choice of α and of the imbeddings h_i.

The geometric idea of the connected sum as two manifolds joined by a tube is visible in the following construction.

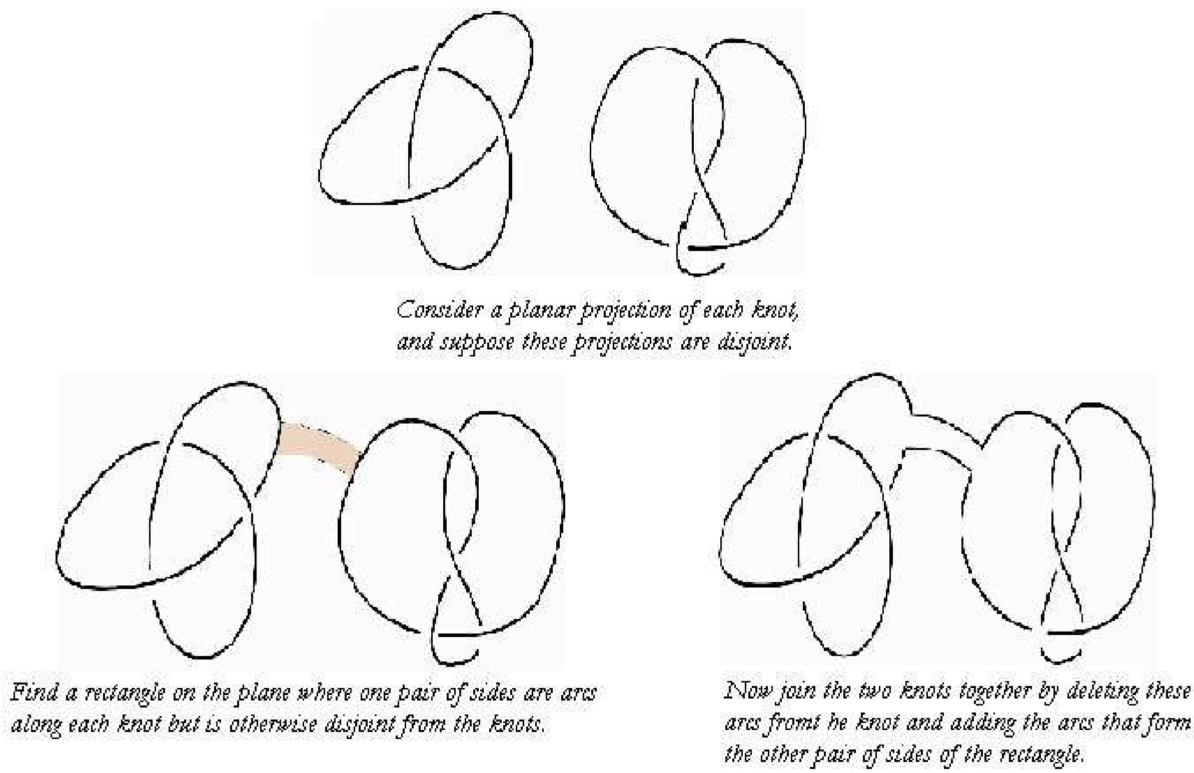

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Figure 1: .

Let $\mathbf{R}_0^m, \mathbf{R}_1^m$ be two copies of $\mathbf{R}^m$ but with opposite orientations and let $h_i : \mathbf{R}^m \to \mathbf{R}_i^m, i = 0, 1$ be imbeddings of $\mathbf{R}^m$ as interior of the unit disc.

Theorem 1. *There is an imbedding h of $\boldsymbol{R}^m \# \boldsymbol{R}^m(h_1, h_2, \alpha)$ in $\boldsymbol{R}^m \times [-1, 1]$ that imbeds $h_1(\boldsymbol{R}^m - 0) \cup h_2(\boldsymbol{R}^m - 0)$ as "a tube" in $\boldsymbol{R}^m \times [-1, 1]$ and such that $h(x) = (x, (-1)^i)$ elsewhere in $\boldsymbol{R}_i^m$.*

Observe now, that if $\mathbf{R}^m \# (-\mathbf{R}^m)$ is so imbedded in $\mathbf{R}^m \times [-1, 1]$, then it bounds a manifold that has $\mathbf{R}^m$ with the interior of a disc deleted as a deformation retract. The deformation simply moves points on *vertical* lines. This has the following consequence, which we will use later. Let M be an oriented manifold, $h : \mathbf{R}^m \to M$ be an imbedding. The connected sum $M\#(-M)$ can now be imbedded in $M \times [-1, 1]$ by imbedding $h(\mathbf{R}^m)\#h(\mathbf{R}^m)$ in $h(\mathbf{R}^m) \times [-1, 1]$ as stated in the above theorem and the rest in the obvious way. The resulting manifold will bound a manifold that has M with the interior of a disc deleted as a deformation retract.In particular, if Σ is a homotopy sphere, (i.e.,a $n-$manifold homotopy equivalent to an $n-$sphere) then Σ with the interior of a disc deleted is a contractible manifold. Therefore:

Corollary 1. *If Σ is a homotopy sphere, then $\Sigma\#(-\Sigma)$ bounds a contractible manifold.*

Definition. *A contractible manifold is one that can continuously be shrunk to a point inside the manifold itself.*

For example, an open ball is a contractible manifold. All manifolds homeomorphic to the ball are contractible, too. One can ask whether all contractible manifolds are homeomorphic to a ball. For dimensions 1 and 2, the answer is classical and it is "yes". In dimension 2, for example, it follows from the Riemann mapping theorem. Dimension 3 presents the first counterexample: the $Whitehead\, manifold$.

Theorem 2. *The connected sum of two manifolds is a homotopy sphere if and only if both manifolds are homotopy spheres.*

Theorem 3. *The set of connected, oriented, and closed $m-$dimensional manifolds is, under the operation of connected sum, an associative and commutative monoid with identity.*

Theorem 4. *If $M\#N$ is homeomorphic to S^m, then M is homeomorphic to S^m.*

3 Attaching handles on manifolds

We now define the operation of attaching handles.

Let $h : S^{\lambda-1} \to \partial M^m$ be an imbedding, and let $\bar{h} : T \to M^m$ be an extension of h and a tubular neighbourhood of $h(S^{\lambda-1})$ in M^m. Then the manifold M_1 obtained from $M^m - h(S^{\lambda-1})$ and $D^m - S^{\lambda-1}$ by identifying $x \in T - S^{\lambda-1}$ with $\bar{h}\alpha(x)$ will be referred to as M *with the handle attached along* $h(S^{\lambda-1})$ and denoted, symbolically, $M_1 = M \cup H^\lambda$; $h(S^{\lambda-1})$ will be called the *attaching sphere*.

We will identify $M^m - h(S^{\lambda-1})$ and $D^m - S^{\lambda-1}$ with their images in $M \cup H^\lambda$. In particular, $D^m - S^{\lambda-1}$ (as a subset of M_1) will be called the *handle*, the $\mu-$disc $D^\mu = D^m \cap \mathbf{R}^\mu$ the *belt disc*, and its boundary $S^{\mu-1}$ the *belt sphere*.

Every closed manifold can be built by starting with a disc and consecutively attaching handles. It is, therefore, important to recognize situations in which different sequences of attachments produce the same result. We present here, two results of this type. The first one concerns the order in which handles are attached.

Theorem 5. *If $M_1 = (M \cup H^\mu) \cup H^\lambda$ and $\lambda \leq \mu$, then M_1 can be obtained by first attaching H^λ and then H^μ.*

The second, deeper, result describes the situation when one handle cancels another, that is, when an attachment of two handles of consecutive dimensions to M produces no change in M. It will turn out that this happens when the attaching sphere of the second handle intersects the belt sphere of the first handle transversely in one point.

Theorem 6. *Cancellation Lemma (Smale). Suppose that* $M = (M_1 \cup H^\lambda) \cup H^{\lambda+1}$*, where the attaching sphere of* $H^{\lambda+1}$ *intersects the belt sphere of* H^λ *transversely in one point. Then* M *is diffeomorphic to* M_1*.*

4 Handle Presentation Theorem

The handle presentation theorem of Milnor and Wallace asserts that every manifold can be constructed by successive attachment of handles. This is presented here in terms of elementary cobordisms − a term we will presently define.

Definition. *An* $(n+1)$ *cobordism is a triple,* $\mathscr{C} = \{M, W, N\}$*, where* W *is an* $(n+1)-$*dimensional manifold, whose boundary* $\partial W = M \cup N$ *is the disjoint union of the* $n-$*dimensional manifolds* M *and* N*. In other words, it is a manifold with boundary whose boundary is partitioned in two.*

Elementary cobordisms amount to attaching a handle.

Suppose we are given two cobordisms $\mathscr{C} = \{V_0, W, V_1\}, \mathscr{C}' = \{V_0', W', V_1'\}$ and a diffeomorphism $h : V_1 \to V_0'$. We can join W and W' using h; let $W_1 = W \cup_h W'$. Then $\partial W_1 = V_0 \cup V_1'$ and $\{V_0, W_1, V_1'\}$ is a cobordism, which will be denoted $\mathscr{C} \cup \mathscr{C}'$. If $\mathscr{C}'$ is a trivial cobordism then the result does not depend on h: we then have $\mathscr{C} \cup \mathscr{C}' = \mathscr{C}$.

The fundamental role played by elementary cobordisms is explained by the following

Theorem 7. *Let* $\mathscr{C}$ *be a cobordism. Then* $\mathscr{C} = \mathscr{C}_1 \cup \mathscr{C}_2 \cup \ldots \mathscr{C}_k$*, where the* $\mathscr{C}_i$ *are elementary cobordisms. Moreover, one can assume that* $i < j$ *implies* $\lambda(i) \leq \lambda(j)$*, where* $\lambda(i)$ *denotes the index of* $\mathscr{C}_i$*.*

The main theorem of the theory developed so far asserts the existence of a presentation with minimal number of handles.

Let $\mathscr{C} = V_0, W, V_1$ be a cobordism. We say that $\mathscr{C}$ is simply connected if V_0, W, V_1 are connected and simply connected. The dimension of $\mathscr{C}$ is by definition the dimension of w. Recall that, for a given presentation, $c_\lambda = \#$ of $\lambda-$handles and $b_\lambda(M, V_0) =$rank $H_\lambda(M, V_0)$.

Theorem 8. *Let* $\mathscr{C}$ *be a simply connected cobordism of dimension* $m \geq 6$ *such that* $H_i(W, V_0)$ *and* $H_i(W, V_1)$ *are free for* $i < k, k > 1$*. Then there is a presentation of* $\mathscr{C}$ *such that* $c_i = b_i(M, V_0)$ *for* $i < k$ *and* $i > m - k$*.*

The presentation obtained, is in fact, minimal. Observe that the hypothesis of the theorem are satisfied if $H_i(W)$ is free for $i < k, k > 1$, and both V_0 and V_1 are $(k-1)$-connected. This is certainly true if W is closed:

Corollary 2. *If W is a simply connecetd closed manifold of dimension $m \geq 6$ and $H_*(W)$ is free, then there is a handle presentation of W such that $c_\lambda = b_\lambda(W), \lambda = 0, 1, \ldots, m$.*

As another corollary of the above theorem we obtain the result:

Theorem 9. (The h-cobordism Theorem) *If $\mathscr{C}$ is a simply connected cobordism of dimension $m \geq 6$ such that $H_*(W, V_0) = 0$, then $\mathscr{C}$ is a trivial cobordism, i.e., W is diffeomorphic to $V_0 \times I$.*

The following result,known as the Disc Bundle Theorem, takes us to within one step from our goal.

Theorem 10. *Let W and ∂W be simply connected and let M be a simply connected closed submanifold in the interior of W. If dim $M + 3 \leq$ dim $W \geq 6$ and $H_*(W, M) = 0$, then W is diffeomorphic to a closed tubular neighborhood T of M in W.*

This yields the following characterization of D^m (take as m a point in the interior of W): If W is contractible with a simply connected boundary and of dimension $m \geq 6$, then W is diffeomorphic to D^m. In particular, there is a smooth structure on D^m.

If M is a homotopy sphere of dimension $m \geq 5$, $M\#(-M)$ bounds a contractible manifold of dimension ≥ 6, thus by the characterization obtained above, is diffeomorphic to D^{m+1}. Hence, M is homeomorphic to S^m.

This establishes the Poincare conjecture for smooth manifolds of dimension larger than 4:

Theorem 11. *If M is a homotopy sphere of dimension $m \geq 5$, then M is homeomorphic to S^m.*

5 References

1. Kosinski, Antoni A., *Differential Manifolds* , Academic Press, 1993.

2. Milnor, J., *Topology from the Differentiable Viewpoint* , The University Press of Virginia, Charlottesville, Virginia, 1965.

3. Smale, S., *On the Structure of Manifolds* , Amer. J. Math. **8**4 (1962).