A REMARK ON DIFFERENTIABLE STRUCTURES

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Yesterday in class I said that the unit square C in $\mathbb{R}^2$ is *not* a smooth manifold. This statement is ambiguous and basically wrong. It's ambiguous because it is not clear if it means that C does not admit any differentiable structures that make it a smooth manifold, or that C is not a smooth manifold with respect to a particular differentiable structure. So let's make things more precise.

Recall first that a differentiable structure is a maximal atlas, but that every atlas is contained in a unique maximal one. So to equip a manifold with a differentiable structure, it is enough to give it an atlas.

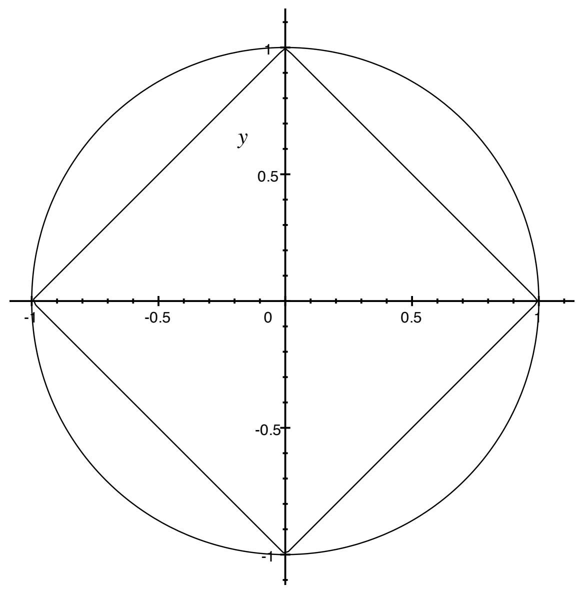

FIGURE 1. C and S^1.

Let us think of C as the set of points (x, y) such that $|x| + |y| = 1$. See Figure **??**. This makes C the unit circle with respect to the norm $\|(x, y)\|_1 = |x| + |y|$ on $\mathbb{R}^2$. Just as in the case of the standard unit circle S^1 with respect to the Euclidean norm $\|\cdot\|_2$, the most natural candidate for an atlas for C is the collection $\mathscr{U} = \{(U_i^{\pm}, \phi_i^{\pm}) : i = 1, 2\}$, where

$$U_i^+ = \{(x_1, x_2) \in C : x_i > 0\}, \qquad U_i^- = \{(x_1, x_2) \in C : x_i < 0\},$$

and

$$\phi_i^{\pm}(x_1, x_2) = x_i.$$

Each $(U_i^{\pm}, \phi_i^{\pm})$ is certainly a chart for C considered as a topological manifold; i.e., each $U_i^{\pm}$ is an open set in C and $\phi_i^{\pm} : U_i^{\pm} \to (-1, 1)$ is a homeomorphism. The collection $\mathscr{U}$ clearly covers C. However,

$$\phi_1^+ \circ (\phi_2^+)^{-1}(t) = 1 - |t|,$$

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(check this!), so this change of coordinates is not smooth. This means that $\mathscr{U}$ is not an atlas for C, hence $(C, \mathscr{U})$ (or to be more precise, $(C, \mathscr{U}_{\max})$, where $\mathscr{U}_{\max}$ is the maximal atlas containing $\mathscr{U}$) is not a smooth manifold.

But this does not mean that C cannot be equipped by a differentiable structure that makes it into a smooth manifold!

In fact, C is homeomorphic to the unit circle S^1 via a homeomorphism $h : C \to S^1$ given by

$$h(p) = \frac{p}{\|p\|_2}.$$

It is not hard to check (check it!) that

$$h^{-1}(x, y) = \left(\frac{x}{|x| + |y|}, \frac{y}{|x| + |y|} \right).$$

Now, S^1 is a smooth manifold relative to the atlas $\mathscr{V} = \{(V_i^\pm, \psi_i^\pm) : i = 1, 2\}$ defined analogously by

$$V_i^+ = \{(x_1, x_2) \in S^1 : x_i > 0\}, \qquad V_i^- = \{(x_1, x_2) \in S^1 : x_i < 0\},$$

and

$$\psi_i^\pm(x_1, x_2) = x_i.$$

Checking that $\mathscr{V}$ is indeed an atlas is completely analogous to how we checked that S^2 is a smooth manifold relative to the atlas defined in class.

We can now define another candidate for an atlas on C by *pulling back* $\mathscr{V}$ to C via h. Define $\mathscr{W} = \{(W_i^\pm, \chi_i^\pm) : i = 1, 2\}$ by

$$W_i^\pm = h^{-1}(V_i^\pm) = U_i^\pm, \qquad \chi_i^\pm = \psi_i^\pm \circ h.$$

Let us check that these charts are indeed smoothly compatible. We have:

$$\begin{aligned} \chi_i^\pm \circ (\chi_j^\pm)^{-1} &= (\phi_i^\pm \circ h) \circ (\phi_j^\pm \circ h)^{-1} \\ &= \phi_i^\pm \circ h \circ h^{-1} \circ (\phi_j^\pm)^{-1} \\ &= \phi_i^\pm \circ (\phi_j^\pm)^{-1}, \end{aligned}$$

which is smooth, by construction (since $(S^1, \mathscr{V})$ is a smooth manifold).

Out of curiosity let us compute χ_2^+ and $(\chi_2^+)^{-1}$:

$$\chi_2^+(x, y) = \psi_2^+ \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right) = \frac{x}{\sqrt{x^2 + y^2}},$$

and

$$(\chi_2^+)^{-1}(t) = h^{-1}(t, \sqrt{1 - t^2}) = \left(\frac{t}{t + \sqrt{1 - t^2}}, \frac{\sqrt{1 - t^2}}{t + \sqrt{1 - t^2}} \right).$$

Of course, this is not the only way to equip C with a differentiable structure. Can you think of another one?

This leads us to the following general result:

Theorem 1. *If M is a topological space, N a smooth manifold, and $h : M \to N$ a homeomorphism, then h defines a differentiable structure on M. This structure is given by*

$$h^*(\mathscr{U}) = \{(h^{-1}(U_i), \phi_i \circ h) : i \in I\}$$

where $\mathscr{U} = \{(U_i, \phi_i) : i \in I\}$ is a differentiable structure on N.

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