San José State University

Math 213, Fall 2007

Final Exam

December 10, 2007

Due on December 17, 2007 by 2 PM

Happy Holidays!

Name:

	Score
1	
2	
3	
4	
5	
6	
Total	

You are allowed to use the literature but not talk to each other.

1. (25 points) Let X, Y be smooth manifolds and Z a regular submanifold of Y. A smooth map $f : X \to Y$ is said to be transverse to Z if for every $p \in f^{-1}(Z)$,

$$\text{image}(T_p f) + T_q Z = T_q Y,$$

where $q = f(p)$. Show that if f is transverse to Z, then the preimage $f^{-1}(Z)$ is a regular submanifold of X whose codimension equals the codimension of Z in Y. Recall that the codimension of Z in Y is $\dim Y - \dim Z$.

Proof:

2. (25 points) (a) Show that S^n is diffeomorphic to the quotient (homogeneous) manifold $SO(n+1)/SO(n)$.

(b) Show that the real projective space $P^n(\mathbb{R})$ is diffeomorphic to $SO(n+1)/O(n)$.

Proof:

3. (25 points) A smooth vector field X with local flow $\{\phi_t\}$ on a Riemannian manifold M is called a Killing vector field if ϕ_t is an isometry, for each t.

(a) Characterize all Killing vector fields when M is the Euclidean space $\mathbb{E}^n$.

(b) Let X be a Killing vector field on M, $p \in M$, and U is a normal neighborhood of p in M. Assume that p is the unique point in U such that $X(p) = 0$. Show that in U, X is tangent to the geodesic spheres $\{\mathrm{Exp}_p(v) : \|v\| = \text{constant}\}$ centered at p.

(c) Suppose $f : M \to N$ is an isometry. Show that X is a Killing vector field on M iff $Y = f_*(X)$ is a Killing vector field on N.

Proof:

4. (25 points) (a) The gradient of a smooth function $f : M \to \mathbb{R}$ on a Riemannian manifold M is a vector field $\operatorname{grad} f$ on M defined by

$$\langle \operatorname{grad} f, X \rangle = Xf,$$

for every smooth vector field X on M, where $\langle \cdot, \cdot \rangle$ denotes the Riemannian metric on M. Find the expression for $\operatorname{grad} f$ in local coordinates.

(b) If X is a smooth vector field on M, the divergence of X is a function $\operatorname{div} X : M \to \mathbb{R}$ defined by:

$$\operatorname{div} X(p) = \operatorname{trace} L_p,$$

where $L_p : T_pM \to T_pM$ is the linear map defined by

$$L_p(v) = (\nabla_v X)(p).$$

Here ∇ denotes the Riemannian connection of M.

If $\{E_1, \ldots, E_n\}$ is a geodesic frame, i.e., a local frame such that $\nabla_{E_i} E_j = 0$, for all $1 \le i, j \le n$, show that

$$X = \sum_{i=1}^{n} a_i E_i \quad \Rightarrow \quad \operatorname{div} X = \sum_{i=1}^{n} E_i(a_i).$$

Proof:

5. (25 points) Let $f : M \to \mathbb{R}$ have p as a critical point, i.e., $T_p f = 0$. Define the Hessian of f at p, $f_{**} : T_pM \times T_pM, \to \mathbb{R}$ as follows. Given $u, v \in T_pM$, choose vector fields X, Y with $X(p) = u$ and $Y(p) = v$, and set

$$f_{**}(u, v) = X_p(Yf).$$

(a) Show that $f_{**}(u, v)$ is well-defined, symmetric, and bilinear.

(b) If $\{E_1, \ldots, E_n\}$ is the coordinate frame corresponding to some local coordinates (U, φ), show that

$$f_{**}\left(\sum_{i=1}^{n} a_i E_i, \sum_{j=1}^{n} b_j E_j\right) = \sum_{i,j=1}^{n} a_i b_j \frac{\partial^2 \tilde{f}}{\partial x_i \partial x_j}(p).$$

where $\tilde{f} = f \circ \varphi^{-1}$ is the representation of f in (U, φ).

(c) Show that the rank of the matrix

$$\left[\frac{\partial^2 \tilde{f}}{\partial x_i \partial x_j}(p)\right]$$

is independent of the coordinate system.

Proof:

6. (35 points) (a) Let G, H be Lie groups. Denote their Lie algebras by $\mathfrak{g}, \mathfrak{h}$ and their exponential maps by $\exp_G, \exp_H$, respectively. If $\phi : G \to H$ is a Lie group homomorphism, show that

$$\phi \circ \exp_G = \exp_H \circ \phi_*,$$

where ϕ_* is the tangent map to ϕ at the identity element of G.

(b) Recall that the adjoint representation of G,

$$\mathrm{Ad} : G \to Gl(\mathfrak{g}),$$

is defined by

$$\mathrm{Ad}(g) = T_e(L_g R_g^{-1}) : \mathfrak{g} \to \mathfrak{g},$$

for each $g \in G$, where e is the identity of G and $Gl(\mathfrak{g})$ denotes the group of linear automorphisms of $\mathfrak{g}$. Define

$$\mathrm{ad} = T_e \, \mathrm{Ad}\,.$$

Show that $\mathrm{Ad} \circ \exp = \exp \circ \, \mathrm{ad}$.

(c) (**Extra credit: 10 points**) Show that $\mathrm{ad}(X)(Y) = [X, Y]$, for all $X, Y \in \mathfrak{g}$.

Proof: