San José State University

Math 213, Spring 2009

Final Exam

Assigned on May 11, 2009

Due on May 18, 2009 by 2 PM

Have a great summer break!

Name:

	Score
1	
2	
3	
4	
5	
6	
XC	
Total	

You are allowed to use the literature but not talk to each other.

1. (25 points) Let $P : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous polynomial of degree k. This means

$$P(tx_1, \ldots, tx_n) = t^k P(x_1, \ldots, x_n),$$

for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

(a) Prove Euler's identity:

$$\sum_{i=1}^{n} x_i \frac{\partial P}{\partial x_i} = kP.$$

(b) Let $M_a = \{x : P(x) = a\}$. Prove that if $a \neq 0$, then M_a is an $(n-1)$-dimensional regular submanifold of $\mathbb{R}^n$.

(c) Show that the manifolds M_a obtained with $a > 0$ are all diffeomorphic, as are those with $a < 0$.

Proof:

2. (25 points) Let $f : M \to N$ be a smooth map. A point $y \in N$ is called a regular value of f if $T_x f : T_x M \to T_y N$ is surjective for every $x \in f^{-1}(y)$. Suppose that M is compact, $\dim M = \dim N$, and let y be a regular value of f.

(a) Show that $f^{-1}(y)$ is a finite set $\{x_1, \ldots, x_N\}$.

(b) Show that there exists a neighborhood V of y such that $f^{-1}(V)$ is a disjoint union $U_1 \cup \ldots \cup U_N$, where U_i is a neighborhood of x_i and f maps U_i diffeomorphically onto V.

Proof:

3. (25 points) Let X be a smooth vector field on a compact manifold M and let $u : M \to \mathbb{R}$ be a smooth strictly positive function. Define a vector field on M by

$$Y(p) = u(p)X(p),$$

for all $p \in M$.

(a) Show that X and Y have the same orbits.

(b) Denote the flows of X and Y by $\{\phi_t\}$ and $\{\psi_t\}$ respectively. Show that there exists a smooth function $\tau : M \times \mathbb{R} \to \mathbb{R}$ such that

$$\psi_t(x) = \phi_{\tau(x,t)}(x),$$

for all $x \in M$ and $t \in \mathbb{R}$. Compute the function τ in terms of u.

Solution:

4. (25 points) Let M be a smooth manifold and $p \in M$. The cotangent space of M at p is the vector space

$$T_p^*M = \{\alpha \mid \alpha : T_pM \to \mathbb{R} \text{ is linear}\}.$$

That is, T_p^*M is the dual of T_pM: $T_p^*M = (T_pM)^*$. The cotangent bundle of M is the union

$$T^*M = \bigcup_{p \in M} T_p^*M.$$

A (differential) 1-form on M is a map $\alpha : M \to T^*M$ such that $\alpha_p \in T_p^*M$, for all $p \in M$. A 1-form α is smooth if the function $p \mapsto \alpha_p(X(p))$ is smooth, for every smooth vector filed X on M.

Prove that the tangent bundle of M is trivial if and only if there exist smooth 1-forms $\alpha^1, \ldots, \alpha^n$ ($n = \dim M$) such that for every $p \in M$, $\alpha_p^1, \ldots, \alpha_p^n$ is a basis for T_p^*M. (In other words, TM is trivial iff T^*M is trivial.)

Proof:

5. (25 points) Let $Sl(n, \mathbb{R})$ be the set of all matrices with determinant $+1$. Show that $Sl(n, \mathbb{R})$ is a submanifold of $\mathscr{M}_n(\mathbb{R})$ and find its dimension.

Solution:

6. (35 points) Let M be a Riemannian manifold. For a C^1 curve γ in M, denote by $L(\gamma)$ its arclength. Show that there exists no smooth 1-form (see problem 4) α such that for every C^1 curve γ in M,

$$\int_\gamma \alpha = L(\gamma).$$

Here, the integral of α over a C^1 curve $\gamma : [a, b] \to M$ is defined as in calculus by

$$\int_\gamma \alpha = \int_a^b \alpha_{\gamma(t)}(\dot{\gamma}(t))\, dt.$$

Proof:

Extra credit (20 points) Let M be a connected complete Riemannian manifold and let $f, g : M \to M$ be isometries. If there exits a point $p \in M$ such that $f(p) = g(p)$ and $T_p f = T_p g$, show that $f = g$.