San José State University

Math 213, Spring 2009

Final Exam Solutions

Assigned on May 11, 2009

Due on May 18, 2009 by 2 PM

Have a great summer break!

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	Score
1	25
2	25
3	25
4	25
5	25
6	25
XC	20
Total	170

You are allowed to use the literature but not talk to each other.

1. (25 points) Let $P : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous polynomial of degree k. This means

$$P(tx_1, \ldots, tx_n) = t^k P(x_1, \ldots, x_n), \tag{1}$$

for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

(a) Prove Euler's identity:

$$\sum_{i=1}^{n} x_i \frac{\partial P}{\partial x_i} = kP.$$

(b) Let $M_a = \{x : P(x) = a\}$. Prove that if $a \neq 0$, then M_a is an $(n-1)$-dimensional regular submanifold of $\mathbb{R}^n$.

(c) Show that the manifolds M_a obtained with $a > 0$ are all diffeomorphic, as are those with $a < 0$.

Proof: (a) Differentiating (1) with respect to t at $t = 1$ and using the chain rule, we obtain Euler's identity.

(b) Suppose $a > 0$ and M_a is nonempty. Let $x \in M_a$ be arbitrary. If x is a critical point of P, then the left-hand side of Euler's identity equals zero, hence $P(x) = 0$, which contradicts the assumption that $P(x) = a$. Therefore, x is a regular point of P. Since $x \in M_a$ was arbitrary, it follows that a is a regular value of P (see problem 2).

Lemma 1 *If $f : M \to N$ is a smooth map and $y \in f(M)$ is a regular value, then $f^{-1}(y)$ is a regular submanifold of M of dimension $\dim M - \dim N$.*

As discussed in class, the lemma follows from the Rank Theorem.

Lemma 1 implies that M_a is a regular submanifold of $\mathbb{R}^n$ of codimension one (i.e., $\dim M_a = n - 1$).

(c) Suppose $a, b > 0$ and M_a is non-empty. Let $t = (b/a)^{1/k}$ and define a map $L : \mathbb{R}^n \to \mathbb{R}^n$ by

$$L(x) = tx.$$

Since $P(tx) = t^k P(x)$, it follows that $P(x) = a$ iff $P(tx) = b$. Since L is clearly a diffeomorphism (it's a linear isomorphism), it follows that L maps M_a diffeomorphically onto M_b. The proof for $a, b < 0$ is similar.

2. (25 points) Let $f : M \to N$ be a smooth map. A point $y \in N$ is called a regular value of f if $T_x f : T_x M \to T_y N$ is surjective for every $x \in f^{-1}(y)$. Suppose that M is compact, $\dim M = \dim N$, and let y be a regular value of f.

(a) Show that $f^{-1}(y)$ is a finite set $\{x_1, \ldots, x_N\}$.

(b) Show that there exists a neighborhood V of y such that $f^{-1}(V)$ is a disjoint union $U_1 \cup \ldots \cup U_N$, where U_i is a neighborhood of x_i and f maps U_i diffeomorphically onto V.

Proof: (a) By Lemma 1, $f^{-1}(y)$ is a regular submanifold of M of dimension zero, i.e., $f^{-1}(y)$ is a discrete subset of M.[1] This implies that $f^{-1}(y)$ is at most countable. If $f^{-1}(y)$ is infinite, then by compactness of M, $f^{-1}(y)$ has an accumulation point, which must be in $f^{-1}(y)$, since $f^{-1}(y)$ is closed. But this contradicts the fact that $f^{-1}(y)$ is discrete. Therefore, $f^{-1}(y)$ is a finite set $\{x_1, \ldots, x_N\}$.

(b) By the Rank Theorem, for each $1 \leq i \leq N$, there exist charts (A_i, ϕ_i) at x_i and (B_i, ψ_i) at y such that $\psi_i \circ f \circ \phi_i^{-1}$ is the identity map on $\mathbb{R}^n$, where $n = \dim M = \dim N$. Since ϕ_i and ψ_i are diffeomorphisms, this implies that $f : A_i \to B_i$ is also a diffeomorphism. But $f^{-1}(y)$ is a finite set, so we can shrink the A_i's if necessary in order to make them pairwise disjoint. So assume A_i's are pairwise disjoint, let $f(A_i) = B_i$ and set

$$V = \bigcap_{i=1}^{N} B_i.$$

Clearly, V is open. Denote by U_i the component of $f^{-1}(V)$ containing x_i. Then U_i is open and $U_1, \ldots, U_N$ are pairwise disjoint, since $U_i \subset A_i$. Furthermore, the restriction of f to U_i maps U_i diffeomorphically onto V. This completes the proof.

This result is called the Stack of Records Theorem. The sets $U_1, \ldots, U_N$ look like a stack of records.

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[1] This means that each $x \in f^{-1}(y)$ has a neighborhood which does not contain any other element of $f^{-1}(y)$.

3. (25 points) Let X be a smooth vector field on a compact manifold M and let $u : M \to \mathbb{R}$ be a smooth strictly positive function. Define a vector field on M by

$$Y(p) = u(p)X(p),$$

for all $p \in M$.

(a) Show that X and Y have the same orbits.

(b) Denote the flows of X and Y by $\{\phi_t\}$ and $\{\psi_t\}$ respectively. Show that there exists a smooth function $\tau : M \times \mathbb{R} \to \mathbb{R}$ such that

$$\psi_t(x) = \phi_{\tau(x,t)}(x),$$

for all $x \in M$ and $t \in \mathbb{R}$. Compute the function τ in terms of u.

Solution: (a) Let $x \in M$ be an arbitrary point. Denote by α and β the integral curves of X and Y, respectively, satisfying $\alpha(0) = \beta(0) = x$. We would like to show that β is a reparametrization of α, i.e., $\beta = \alpha \circ \varrho$, for some diffeomorphism $\varrho : \mathbb{R} \to \mathbb{R}$. We will construct $\varrho = \rho_x$ as a solution to a differential equation as follows.

Since u and α are smooth, there exists a unique function $\rho_x : \mathbb{R} \to \mathbb{R}$ satisfying the differential equation

$$\rho_x'(t) = (u \circ \alpha)(\rho_x((t))), \qquad \rho_x(0) = 0.$$

Define

$$\gamma(t) = \alpha(\rho_x(t)).$$

Then $\gamma : \mathbb{R} \to M$ is smooth, $\gamma(0) = \alpha(0) = x$ and

$$\begin{aligned} \dot\gamma(t) &= \dot\rho_x(t)\ \dot\alpha(\rho_x(t)) \\ &= u(\alpha(\rho_x(t)))X(\alpha(\rho_x(t))) \\ &= Y(\gamma(t)). \end{aligned}$$

Therefore, γ is an integral curve of Y satisfying $\gamma(0) = 0$. By uniqueness, $\gamma = \beta$, which means

$$\beta = \alpha \circ \rho_x.$$

Observe that since $\dot\rho_x(t) \geq \min u > 0$, $\rho_x : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism, so the sets $\alpha(\mathbb{R})$ and $\beta(\mathbb{R})$ are the same. This means that the orbits of X and Y through x coincide.

(b) By (a) τ exists and we can take $\tau(x,t) = \rho_x(t)$. Alternatively, differentiating $\psi_t(x) = \phi_{\tau(x,t)}(x)$ with respect to t, we obtain

$$Y(\psi_t(x)) = \dot\tau(x,t)X(\phi_{\tau(x,t)}(x)),$$

which implies $\dot\tau(x,t) = u(\psi_t(x))$. Integrating and using $\tau(x,0) = 0$, we obtain

$$\tau(x,t) = \int_0^t u(\psi_s(x))\, ds.$$

4. **(25 points)** Let M be a smooth manifold and $p \in M$. The cotangent space of M at p is the vector space

$$T_p^*M = \{\alpha \mid \alpha : T_pM \to \mathbb{R} \text{ is linear}\}.$$

That is, T_p^*M is the dual of T_pM: $T_p^*M = (T_pM)^*$. The cotangent bundle of M is the union

$$T^*M = \bigcup_{p \in M} T_p^*M.$$

A (differential) 1-form on M is a map $\alpha : M \to T^*M$ such that $\alpha_p \in T_p^*M$, for all $p \in M$. A 1-form α is smooth if the function $p \mapsto \alpha_p(X(p))$ is smooth, for every smooth vector filed X on M.

Prove that the tangent bundle of M is trivial if and only if there exist smooth 1-forms $\alpha^1, \ldots, \alpha^n$ ($n = \dim M$) such that for every $p \in M$, $\alpha_p^1, \ldots, \alpha_p^n$ is a basis for T_p^*M. (In other words, TM is trivial iff T^*M is trivial.)

Proof: We will use the following lemma.

Lemma 2 *Let V be a vector space If $v_1, \ldots, v_n$ is a basis for V, then $v_1^*, \ldots, v_n^*$ is a basis of the dual space V^*, where v_i^* is defined by $v_i^*(v_j) = \delta_{ij}$ (the Kronecker delta).*

Proof: Let $v^* \in V^*$ be arbitrary. Define $\alpha_i = v^*(v_i)$, $1 \leq i \leq n$. We claim that

$$v^* = \sum_{i=1}^{n} \alpha_i v_i^*.$$

Denote the right-hand side by w^*. Then for all $1 \leq j \leq n$, $v^*(v_j) = \alpha_j = w^*(v_j)$. Since v^* and w^* agree on a basis of V, $v^* = w^*$. This completes the proof of the lemma.

($\Rightarrow$) Suppose TM is trivial. Then there exist smooth vector fields $X_1, \ldots, X_n$ ($n = \dim M$) that globally span TM (i.e., $T_pM = \operatorname{span}\{X_1(p), \ldots, X_n(p)\}$, for all $p \in M$). Define 1-forms $\alpha^1, \ldots, \alpha^n$ by

$$\alpha^i(X_j) = \delta_{ij}. \tag{2}$$

Since α^i evaluated on a smooth vector field yields a smooth function, it follows that each α^i is smooth. Let $p \in M$ be arbitrary. Since T_p^*M is the dual to T_pM, Lemma 2 implies that $\alpha_p^1, \ldots, \alpha_p^n$ form a basis of T_p^*M.

($\Leftarrow$) Suppose that T^*M is trivial, i.e., there exist 1-forms $\alpha^1, \ldots, \alpha^n$ that globally span T^*M. Define vector fields $X_1, \ldots, X_n$ by requiring (2). (Observe that each X_i is well-defined, by linear algebra.) Each X_i is smooth, since α^j's are smooth. Let $p \in M$ be arbitrary and recall that T_pM is the dual of T_p^*M.[2] By Lemma 2, it follows that $X_1(p), \ldots, X_n(p)$ form a basis for T_pM.

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[2] $(V^*)^* = V$, for every finite dimensional vector space V.

5. (25 points) Let $Sl(n, \mathbb{R})$ be the set of all matrices with determinant $+1$. Show that $Sl(n, \mathbb{R})$ is a submanifold of $\mathscr{M}_n(\mathbb{R})$ and find its dimension.

Solution: We will identify $\mathscr{M}_n(\mathbb{R})$ with $\mathbb{R}^N$, where $N = n^2$ and use the result of Problem 1(a). Define a function $f : \mathbb{R}^N \to \mathbb{R}$ by

$$f(X) = \det X,$$

where $X \in \mathbb{R}^N$ is identified with the corresponding matrix $X \in \mathscr{M}_n(\mathbb{R})$. Clearly, f is a polynomial in the components of X and by the well-known properties of determinants,

$$f(tX) = t^n f(X),$$

for all X and $t \in \mathbb{R}$. By Problem 1 (a), it follows that $Sl(n, \mathbb{R}) = f^{-1}(1)$ is an $(n^2 - 1)$-dimensional regular submanifold of $\mathbb{R}^N \approx \mathscr{M}_n(\mathbb{R})$.

6. (25 points) Let M be a Riemannian manifold. For a C^1 curve γ in M, denote by $L(\gamma)$ its arclength. Show that there exists no smooth 1-form (see problem 4) α such that for every C^1 curve γ in M,

$$\int_\gamma \alpha = L(\gamma).$$

Here, the integral of α over a C^1 curve $\gamma : [a, b] \to M$ is defined as in calculus by

$$\int_\gamma \alpha = \int_a^b \alpha_{\gamma(t)}(\dot{\gamma}(t))\, dt.$$

Proof: Suppose there exists such a 1-form α. Let $p \in M$ be arbitrary and let

$$K_p = \mathrm{Ker}(\alpha_p) = \{v \in T_pM : \alpha_p(v) = 0\}.$$

Observe that $\dim K_p \in \{n, n-1\}$, where $n = \dim M$.

Take an arbitrary unit speed curve $\gamma : [0, 1] \to M$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) \in K_p$, and denote by γ_t the restriction of γ to the interval $[0, t]$, $0 \leq t \leq 1$. Then by assumption,

$$\begin{aligned} t &= L(\gamma_t) \\ &= \int_{\gamma_t} \alpha \\ &= \int_0^t \alpha_{\gamma(s)}(\dot{\gamma}(s))\, ds, \end{aligned}$$

for all $0 \leq t \leq 1$. Differentiating with respect to t at $t = 0$, we obtain

$$1 = \alpha_p(\dot{\gamma}(0)) = 0,$$

which is a contradiction.

Therefore, no such 1-form α exists.

Extra credit (20 points) Let M be a connected complete Riemannian manifold and let $f, g : M \to M$ be isometries. If there exits a point $p \in M$ such that $f(p) = g(p)$ and $T_p f = T_p g$, show that $f = g$.

Proof: Let $q \in M$ be arbitrary. Since M is a complete and connected Riemannian manifold, the Hopf-Rinow theorem yields a geodesic $\gamma : [0, T] \to M$ connecting p and q. Isometries map geodesics to geodesics, so $\alpha = f(\gamma)$ and $\beta = g(\gamma)$ are both geodesics starting at the same point, $f(p) = g(p)$. Furthermore,

$$\alpha'(0) = T_p f(\gamma'(0)) = T_p g(\gamma(0)) = \beta'(0),$$

so by uniqueness of geodesics, $\alpha = \beta$. Therefore,

$$f(q) = f(\gamma(T)) = g(\gamma(T)) = g(q).$$

Since q was arbitrary, it follows that $f = g$.