

SAN JOSÉ STATE UNIVERSITY

Math 213, Spring 2009

Midterm Exam

ASSIGNED ON MARCH 12, 2009

Due on March 19, 2009 by 2:30 PM

Honor system is in effect.

Name:

	Score
1	
2	
3	
4	
XC	
Total	

All manifolds are assumed to be of class C^∞ .

1. (25 points) By identifying \mathbb{R}^2 with \mathbb{C} , we can think of the unit circle S^1 as a subset of the complex plane. An **angle function** on a subset $U \subset S^1$ is a continuous function $\theta : U \rightarrow \mathbb{R}$ such that $e^{i\theta(p)} = p$, for all $p \in U$.
- (a) Show that there exists an angle function on an open subset $U \subset S^1$ if and only if $U \neq S^1$.
 - (b) For any such angle function θ on U , show that (U, θ) is a smooth structure chart for S^1 with its standard smooth structure.¹

Proof:

¹The standard smooth structure on S^1 is $\{(U_i^\pm, \phi_i^\pm) : i = 1, 2\}$, where $U_i^\pm = \{(x_1, x_2) \in S^1 : \pm x_i > 0\}$ and $\phi_i^\pm(x_1, x_2) = x_j, j \in \{1, 2\} \setminus \{i\}$.

2. (25 points) Let M be a compact smooth manifold. Show that there is no submersion $f : M \rightarrow \mathbb{R}$.

Proof:

3. (25 points) (a) Define the stereographic coordinates on S^1 .

(b) Compute the representation of the n^{th} power map $p_n : S^1 \rightarrow S^1$ ($n \in \mathbb{Z}$),

$$p_n(z) = z^n$$

in stereographic coordinates defined in (a). Show that p_n is smooth for all $n \in \mathbb{Z}$.

Proof:

4. (25 points) We say that m C^1 functions $f_1, \dots, f_m : U \rightarrow \mathbb{R}$ (where $U \subset \mathbb{R}^n$ is an open set) are **dependent** if there exists a C^1 function $\Phi : \mathbb{R}^m \rightarrow \mathbb{R}$ which does *not* vanish on any open set in \mathbb{R}^m but such that

$$\Phi(f_1(x), \dots, f_m(x)) = 0,$$

for all $x \in U$.

Show that if $f_1, \dots, f_m : U \rightarrow \mathbb{R}$ are C^1 functions and the rank of

$$F(x) = (f_1(x), \dots, f_m(x))$$

is constant and $< m$ on U , then f_1, \dots, f_m are (locally) dependent.

Proof:

Extra credit (20 points) Show that \mathbb{R} has uncountably many distinct (i.e., smoothly incompatible) smooth structures.

Proof: