All manifolds are assumed to be of class $C^\infty$. 
1. **(25 points)** By identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can think of the unit circle $S^1$ as a subset of the complex plane. An angle function on a subset $U \subset S^1$ is a continuous function $\theta : U \to \mathbb{R}$ such that $e^{i\theta(p)} = p$, for all $p \in U$.

(a) Show that there exists an angle function on an open subset $U \subset S^1$ if and only if $U \neq S^1$.

(b) For any such angle function $\theta$ on $U$, show that $(U, \theta)$ is a smooth structure chart for $S^1$ with its standard smooth structure. \(^1\)

**Proof:**

\(^1\)The standard smooth structure on $S^1$ is $\{(U^+_i, \phi_i^+) : i = 1, 2\}$, where $U^+_i = \{(x_1, x_2) \in S^1 : \pm x_i > 0\}$ and $\phi_i^+(x_1, x_2) = x_j$, $j \in \{1, 2\} \setminus \{i\}$. 

2. (25 points) Let $M$ be a compact smooth manifold. Show that there is no submersion $f : M \to \mathbb{R}$.

Proof:
3. (25 points) (a) Define the stereographic coordinates on $S^1$.

(b) Compute the representation of the $n^{th}$ power map $p_n : S^1 \to S^1$ ($n \in \mathbb{Z}$),

$$p_n(z) = z^n$$

in stereographic coordinates defined in (a). Show that $p_n$ is smooth for all $n \in \mathbb{Z}$.

Proof:
4. (25 points) We say that $m$ $C^1$ functions $f_1, \ldots, f_m : U \to \mathbb{R}$ (where $U \subset \mathbb{R}^n$ is an open set) are dependent if there exists a $C^1$ function $\Phi : \mathbb{R}^m \to \mathbb{R}$ which does not vanish on any open set in $\mathbb{R}^m$ but such that

$$
\Phi(f_1(x), \ldots, f_m(x)) = 0,
$$

for all $x \in U$.

Show that if $f_1, \ldots, f_m : U \to \mathbb{R}$ are $C^1$ functions and the rank of

$$
F(x) = (f_1(x), \ldots, f_m(x))
$$

is constant and $< m$ on $U$, then $f_1, \ldots, f_m$ are (locally) dependent.

**Proof:**

...
Extra credit (20 points) Show that \(\mathbb{R}\) has uncountably many distinct (i.e., smoothly incompatible) smooth structures.

Proof: