San José State University

Math 213, Spring 2009

Midterm Exam Solutions

Assigned on March 12, 2009

Due on March 19, 2009 by 2:30 PM

Honor system is in effect.

Name: Granwyth Hultaberi

	Score
1	25
2	25
3	25
4	25
XC	20
Total	120

All manifolds are assumed to be of class C^∞.

1. (25 points) By identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can think of the unit circle S^1 as a subset of the complex plane. An angle function on a subset $U \subset S^1$ is a continuous function $\theta : U \to \mathbb{R}$ such that $e^{i\theta(p)} = p$, for all $p \in U$.

(a) Show that there exists an angle function on an open subset $U \subset S^1$ if and only if $U \neq S^1$.

(b) For any such angle function θ on U, show that (U, θ) is a smooth structure chart for S^1 with its standard smooth structure. [1]

Proof: ($\Rightarrow$) Suppose $\theta : U \to \mathbb{R}$ is an angle function and $U = S^1$. Since θ is continuous and S^1 is compact and connected, $I = \theta(S^1)$ is a compact interval. Furthermore, since $e^{i\theta(p)} = p$, for all $p \in S^1$, it follows that θ is 1–1. By a topology lemma, θ is a homeomorphism from S^1 to I. This is a contradiction (removing one point from I separates it into two components, whereas removing one point from S^1 does not). Therefore, $U \neq S^1$.

($\Leftarrow$) Suppose $U \neq S^1$ is an open subset of S^1. Define $\phi : [0, 2\pi] \to S^1$ by $\phi(t) = e^{it}$. Continuity of ϕ implies that $V = \phi^{-1}(U)$ is open in $[0, 2\pi]$. Since $V \neq [0, 2\pi]$, it follows that ϕ is 1–1 on V. Thus $\phi : V \to U$ is a continuous bijection. Define $\theta = (\phi\restriction_V)^{-1}$ (the inverse of the restriction of θ to V). Then $e^{i\theta(p)} = p$, for all $p \in U$. It remains to show that θ is continuous. Let $J \subset V$ be an open interval. Then

$$\theta^{-1}(J) = \phi(J)$$

is open in S^1. This proves that θ is continuous hence an angle function.

(b) Let $\theta : U \to \mathbb{R}$ be an angle function. It was shown in (a) that θ is a homeomorphism onto its image, hence a chart for S^1. We need to show that it is compatible with the charts in the standard smooth structure on S^1. Assume without loss of generality that $U \cap U_1^+$ is non-empty. Then:

$$\begin{aligned}(\theta \circ (\phi_1^+)^{-1})(t) &= \theta(\sqrt{1-t^2}, t) \\ &= \arctan \frac{t}{\sqrt{1-t^2}},\end{aligned}$$

which is smooth on $\phi_1^+(U \cap U_1^+) \subset (-1, 1)$. Alternatively, $(\phi_1^+ \circ \theta^{-1})(t) = \sin t$. The argument is analogous in all the other cases. Therefore, (U, θ) is smoothly compatible with every chart in the standard smooth structure for S^1.

---

[1] The standard smooth structure on S^1 is $\{(U_i^{\pm}, \phi_i^{\pm}) : i = 1, 2\}$, where $U_i^{\pm} = \{(x_1, x_2) \in S^1 : \pm x_i > 0\}$ and $\phi_i^{\pm}(x_1, x_2) = x_j$, $j \in \{1, 2\} \setminus \{i\}$.

2. (25 points) Let M be a compact smooth manifold. Show that there is no submersion $f : M \to \mathbb{R}$.

Proof: Since M is compact, f achieves its maximum at some point $p \in M$. Let (U, ϕ) be a smooth chart at p and let $\hat{f} = f \circ \phi^{-1}$. Since f has a maximum at p it follows that $\hat{f}$ has a local maximum at $x = \phi(p)$. Therefore, x is a critical point of $\hat{f}$, i.e.,

$$D\hat{f}(x) = \mathbf{0}.$$

This implies that $\mathrm{rank}_p f = \mathrm{rank}_x \hat{f} = 0$, which means that f is not a submersion.

3. (25 points) (a) Define the stereographic coordinates on S^1.

(b) Compute the representation of the n^{th} power map $p_n : S^1 \to S^1$ $(n \in \mathbb{Z})$,

$$p_n(z) = z^n$$

in stereographic coordinates defined in (a). Show that p_n is smooth for all $n \in \mathbb{Z}$.

Proof: (a) Let $N = (0, 1)$ and $S = (0, -1)$ be the north and south pole of S^1. In analogy with S^2 (see homework), the stereographic coordinates on S^1 are (U_N, ϕ_N) and (U_S, ϕ_S), where $U_P = S^1 \setminus \{P\}$ $(P \in \{N, S\})$,

$$\phi_N(x_1, x_2) = \frac{x_1}{1 - x_2}, \qquad \phi_N^{-1}(u) = \left(\frac{2u}{1+u^2}, \frac{u^2-1}{1+u^2}\right),$$

and

$$\phi_S(x_1, x_2) = \frac{x_1}{1 + x_2}, \qquad \phi_S^{-1}(u) = \left(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2}\right).$$

(b) Observe first that if $z = e^{i\theta} = (\cos\theta, \sin\theta)$, for some $\theta \in \mathbb{R}$, then $p_n(z) = (\cos n\theta, \sin n\theta)$.

Assume $z \in U_N$ and $p_n(z) \in U_S$. Then for all $u \in (-1, 1)$, we have:

$$\begin{aligned}
(\phi_S \circ p_n \circ \phi_N^{-1})(u) &= (\phi_S \circ p_n)\left(\frac{2u}{1+u^2}, \frac{u^2-1}{1+u^2}\right) \\
&= (\phi_S \circ p_n)(\cos\theta, \sin\theta) \\
&= \phi_S(\cos n\theta, \sin n\theta) \\
&= \frac{\cos n\theta}{1 + \sin n\theta} \\
&= \frac{\cos(n \cot^{-1} \frac{2u}{u^2-1})}{1 + \sin(n \cot^{-1} \frac{2u}{u^2-1})},
\end{aligned}$$

where $\theta = \cot^{-1} \frac{2u}{u^2-1}$. This map is smooth for all $n \in \mathbb{Z}$ (note that $-1 < u < 1$). All other cases are analogous. Therefore, $p_n : S^1 \to S^1$ is a smooth map.

4. (25 points) We say that m C^1 functions $f_1, \ldots, f_m : U \to \mathbb{R}$ (where $U \subset \mathbb{R}^n$ is an open set) are dependent if there exists a C^1 function $\Phi : \mathbb{R}^m \to \mathbb{R}$ which does *not* vanish on any open set in $\mathbb{R}^m$ but such that

$$\Phi(f_1(x), \ldots, f_m(x)) = 0,$$

for all $x \in U$.

Show that if $f_1, \ldots, f_m : U \to \mathbb{R}$ are C^1 functions and the rank of

$$F(x) = (f_1(x), \ldots, f_m(x))$$

is constant and $< m$ on U, then $f_1, \ldots, f_m$ are (locally) dependent.

Proof: Suppose $\operatorname{rank} F \equiv k < m$ on U. Let $p \in U$ be an arbitrary point and set $q = F(p)$. By the Rank Theorem, there exist smooth charts (V, ϕ) at p and (W, ψ) at q, with $V \subset U$, such that $\hat{F} = \psi \circ F \circ \phi^{-1} : \phi(V) \to \psi(W)$ satisfies

$$\hat{F}(x_1, \ldots, x_n) = (x_1, \ldots, x_k, \overbrace{0, \ldots, 0}^{m-k}).$$

Let ϕ_i be the i^{th} component of ϕ, i.e., assume $\phi(z) = (\phi_1(z), \ldots, \phi_n(z))$, for all $z \in V$. Writing $\phi^{-1}(x) = z$, we obtain $x_i = \phi_i(z)$, so

$$\psi(F(z)) = \psi(F(\phi^{-1}(x))) = (\phi_1(z), \ldots, \phi_k(z), \overbrace{0, \ldots, 0}^{m-k}). \tag{1}$$

Let $\psi = (\psi_1, \ldots, \psi_m)$. Observe that $\psi(F(z)) = (\psi_1(F(z)), \ldots, \psi_m(F(z)))$, so (1) implies $\psi_{k+1}(F(z)) = 0$, for all $z \in V$.

Define

$$\Phi = \psi_{k+1},$$

so that $\Phi(F(z)) = 0$ on V. By definition, Φ is C^1. It remains to show that Φ does not vanish on any open set. Suppose it did, i.e., $\Phi = 0$ on some open set O. Then for $y \in O$,

$$\psi(y) = (\psi_1(y), \ldots, \psi_k(y), 0, \psi_{k+2}(y), \ldots, \psi_m(y)).$$

The $(k+1)^{st}$ row of $D\psi$ vanishes on O, so $\det D\psi(y) = 0$, for all $y \in O$, and ψ is not a diffeomorphism. This is a contradiction. Therefore, Φ does not vanish on any open set, hence $f_1, \ldots, f_m$ are dependent on V.

Extra credit (20 points) Show that $\mathbb{R}$ has uncountably many distinct (i.e., smoothly incompatible) smooth structures.

Proof: For each $\alpha > 0$, set $U_\alpha = \mathbb{R}$ and define $\phi_\alpha : U_\alpha \to \mathbb{R}$ by

$$\phi_\alpha(x) = \begin{cases} x^\alpha, & x \geq 0 \\ -|x|^\alpha, & x < 0. \end{cases}$$

It is not hard to see that ϕ_α is 1–1, onto and continuous. Furthermore, $(\phi_\alpha)^{-1} = \phi_{1/\alpha}$, so ϕ_α is a homeomorphism. Therefore, $\mathscr{U}_\alpha = \{(U_\alpha, \phi_\alpha)\}$ is a smooth atlas for $\mathbb{R}$.

Suppose $\alpha \neq \beta$, e.g., $\alpha > \beta$. Then for $y \geq 0$:

$$(\phi_\beta \circ \phi_\alpha^{-1})(y) = y^{\beta/\alpha},$$

which is not differentiable at 0. Therefore, the charts (U_α, ϕ_α) and (U_β, ϕ_β) are not smoothly compatible.

This shows that $\mathbb{R}$ admits uncountably many[2] smoothly incompatible smooth structures.

Note, however, that $\mathbb{R}$ with the smooth structure given by $\mathscr{U}_\alpha$ is diffeomorphic to $\mathbb{R}$ with its standard smooth structure (given by the identity map). Namely, the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \phi_\alpha$ is a diffeomorphism from $(\mathbb{R}, \mathscr{U}_\alpha)$ to $(\mathbb{R}, \text{standard})$, since

$$\hat{f} = \text{identity} \circ f \circ \phi_\alpha^{-1} = \text{identity}$$

is smooth, as is its inverse.

---

[2] The set $(0, \infty)$ is uncountable.