BASICS OF TOPOLOGY

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Roughly speaking, topology is the area of mathematics that studies the "shape" of spaces. More precisely:

Definition 1. A topology on a set X is a collection $\mathscr{T}$ of subsets of X such that:

(a) the empty set and X are in $\mathscr{T}$;
(b) the union of any subcollection of $\mathscr{T}$ is in $\mathscr{T}$;
(c) the intersection of *finitely* many elements of $\mathscr{T}$ is in $\mathscr{T}$.

Definition 2. A topological space is a pair $(X, \mathscr{T})$, where X is a set and $\mathscr{T}$ is a topology on X.

Definition 3. If $\mathscr{T}$ is a topology on X, then each set $U \in \mathscr{T}$ is called an open set. A set $F \subset X$ is called closed if its complement is open.

Theorem 1. *Suppose $(X, \mathscr{T})$ is a topological space. Then:*

(a) *The empty set and X are closed.*
(b) *The intersection of any collection of closed sets is closed.*
(c) *A* finite *union of closed sets is closed.*

Example 1. If X is any set, then the collection of all subsets of X is a topology on X called the discrete topology.

Example 2. The topology $\mathscr{T} = \{\emptyset, X\}$ is called the trivial (or indiscrete) topology on X.

Example 3. Let $X = \mathbb{R}^n$ and let $d(x, y)$ denote the Euclidean distance between x and y. For $x \in \mathbb{R}^n$ and $r > 0$, denote by

$$B(x, r) = \{y \in \mathbb{R}^n : d(x, y) < r\}$$

the open ball of radius r centered at x. Call a set $U \subset \mathbb{R}^n$ open if for every $x \in U$ there exists $r > 0$ such that $B(x, r) \subset U$. Then the collection of open sets is a topology on $\mathbb{R}^n$ called the Euclidean topology.

Example 4. Let (X, d) be a metric space. This means that X is a set and $d : X \times X \to \mathbb{R}$ a function satisfying the following properties:

(a) $d(x, y) \geq 0$;
(b) $d(x, y) = 0$ if and only if $x = y$;
(c) $d(y, x) = d(x, y)$ (symmetry);
(d) $d(x, y) \leq d(x, z) + d(z, y)$ (the triangle inequality),

for all $x, y, z \in X$. For each $x \in X$ and $r > 0$, denote by

$$B(x, r) = \{y \in X : d(x, y) < r\}$$

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the open ball of radius r centered at x. Call a set $U \subset X$ open if for every $x \in U$ there exists $r > 0$ such that $B(x, r) \subset U$. Then the collection of open sets is a topology on X called the metric space topology.

Definition 4. A topological space $(X, \mathscr{T})$ is called Hausdorff if for every two distinct points $x, y \in X$ there exist disjoint open sets U, V such that $x \in U$ and $y \in V$. That is, every two distinct points can be separated by open sets.

Theorem 2. *Every metric space is Hausdorff.*

Definition 5. A collection $\mathscr{E} = \{E_i : i \in I\}$ of sets is called a cover of E if the union of $\mathscr{E}$ contains E, i.e.,

$$E \subset \bigcup_{i \in I} E_i.$$

A cover $\{E_i\}$ is called open if each E_i is an open set. It is called finite if I is finite.

Definition 6. A subset K of a topological space X is called compact if every open cover of K has a finite subcover. That is, for every open cover $\mathscr{U} = \{U_i : i \in I\}$ of K there is a finite subcollection $\{U_{i_1}, \ldots, U_{i_k}\} \subset \mathscr{U}$ such that

$$K \subset U_{i_1} \cup \cdots \cup U_{i_k}.$$

Theorem 3 (Heine-Borel)**.** *In $\mathbb{R}^n$ a set is compact if and only if it is closed and bounded.*[1]

Theorem 4. *A set $K \subset \mathbb{R}^n$ is compact if and only if every sequence in K has a convergent subsequence. That is, for every sequence (x_k) there exists a subsequence (x_{k_i}) which converges to an element of K.*

Definition 7. A topological space X is disconnected if it is the union of two open, nonempty disjoint sets. Otherwise, it is called connected.

Example 5. $\mathbb{R}^n$ is connected for every n. $\{0, 1\}$ is disconnected.

Theorem 5. *The following statements are equivalent:*

(a) *X is connected.*
(b) *The only subsets of X that are both open and closed are $\emptyset$ and X.*
(c) *Every continuous function $f : X \to \{0, 1\}$ is constant.*

Definition 8. A topological space $(X_0, \mathscr{T}_0)$ is called a (topological) subspace of $(X, \mathscr{T})$ if $X_0 \subset X$ and for every $U \in \mathscr{T}_0$ there exists $V \in \mathscr{T}$ such that $U = V \cap X_0$.

Example 6. If $m < n$, then $\mathbb{R}^m$ is a subspace of $\mathbb{R}^n$, where $\mathbb{R}^m$ is identified with $\mathbb{R}^m \times \{(0, \ldots, 0)\}$ ($n - m$ zeros).

Definition 9. A collection $\mathscr{B}$ of open subsets of a topological space X is called a basis for a topology on X if:

(a) $\mathscr{B}$ covers X;
(b) if $x \in B_1 \cap B_2$, for some $B_1, B_2 \in \mathscr{B}$, then there exists $B \in \mathscr{B}$ such that $x \in B$ and $B \subset B_1 \cap B_2$.

Definition 10. If $\mathscr{B}$ is a basis for a topology on X, then the topology $\mathscr{T}$ generated by $\mathscr{B}$ is defined as follows: $U \subset X$ is said to be open (i.e., $U \in \mathscr{T}$) if for each $x \in U$ there exists $B \in \mathscr{B}$ such that $x \in B$ and $B \subset U$.

[1] That is, contained in a ball $B(\mathbf{0}, r)$, for some $r > 0$.

Definition 11. A topological space is called second countable if it has a countable basis.

Example 7. $\mathbb{R}^n$ is second countable. Let $\{q_1, q_2, \ldots\}$ be an enumeration of all points with rational coordinates in $\mathbb{R}^n$ and let $\{r_1, r_2, \ldots\}$ be an enumeration of all positive rational numbers. Then the collection of open balls $B(q_i, r_j)$ forms a basis for the Euclidean topology on $\mathbb{R}^n$ and is countable.

Definition 12. A map $f : X \to Y$ between topological spaces is continuous if for every set V open in Y, its *preimage* $f^{-1}(V) = \{x \in X : f(x) \in V\}$ is open in X.

Theorem 6. *Suppose $f : X \to Y$ is continuous. If $K \subset X$ is compact, then $f(K) = \{f(x) : x \in K\}$ is compact. If $C \subset X$ is connected, then $f(C)$ is connected.*

Definition 13. A continuous bijection $f : X \to Y$ whose inverse is also continuous is called a homeomorphism. If $f : X \to Y$ is a homeomorphism, then X and Y are called homeomorphic.

Example 8. Any open interval (a, b) is homeomorphic to $\mathbb{R}$. The open unit disk D in $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^2$.

Example 9. The map $\phi : [0, 2\pi) \to S^1$, where S^1 is the unit circle, defined by $\phi(t) = (\cos t, \sin t)$ is a continuous bijection, but its inverse is not continuous. Thus ϕ is *not* a homeomorphism. This can also be seen as follows: if ϕ^{-1} were continuous, then $\phi^{-1}(S^1) = [0, 2\pi)$ would be compact, which is not the case.

Theorem 7 (Invariance of domain)**.** *If $m \neq n$, then $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$.*

Let X be a topological space and $\sim$ an equivalence relation on X. Denote by $\pi : X \to X/\sim$ the quotient map,

$$\pi(x) = [x],$$

where $[x]$ denotes the equivalence class of x. Declare a set $V \subset X/\sim$ to be open in $X/\sim$ if and only if its preimage $\pi^{-1}(U)$ is open in X. This defines a topology on $X/\sim$ called the quotient topology.

Definition 14. The space $X/\sim$ equipped with the quotient topology is called the quotient space of X (relative to $\sim$).

Example 10. Let $X = \mathbb{R}$ and define $x \sim y$ if and only if $x - y$ is an integer. Then $\sim$ is an equivalence relation and $\mathbb{R}/\sim$ is homeomorphic to the unit circle S^1.

Theorem 8. *We have:*

(a) $\pi : X \to X/\sim$ *is continuous.*
(b) *Suppose the maps $f : X/\sim \to Y$ and $g : X \to Y$ satisfy $g = f \circ \pi$. Then f is continuous if and only if g is continuous.*

For the "rest" of general topology, you can check the following references:

- K. Janich, *Topology*, Springer, 1984;
- J. Munkres, *Topology*, second edition, Prentice Hall, 2000

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