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Explain your work
1. (25 points) Consider a surface patch

\[ \sigma(u, v) = (u \cos v, u \sin v, \log \cos v + u), \]

where \( u \in \mathbb{R} \) and \(-\frac{\pi}{2} < v < \frac{\pi}{2}\).

(a) Compute the first fundamental form of \( \sigma \).

(b) Let \( v_1, v_2 \) be any two numbers in \((-\frac{\pi}{2}, \frac{\pi}{2})\) and define curves

\[ \gamma_i(t) = \sigma(t, v_i), \quad i = 1, 2, \quad a \leq t \leq b, \]

where \( a < b \) are arbitrary. Show that \( \gamma_1 \) and \( \gamma_2 \) have the same arclength.

**Solution:** Since

\[ \sigma_u = (\cos v, \sin v, 1), \quad \sigma_v = (-u \sin v, u \cos v, -\tan v), \]

it follows that

\[ E = \|\sigma_u\|^2 = 2, \quad F = \sigma_u \cdot \sigma_v = -\tan v, \quad G = \|\sigma_v\| = u^2 + \tan^2 v. \]

(b) Recall that \( \|\dot{\gamma}_i\|^2 = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2 \). Since \( u = t \) and \( v = v_i \) for the curve \( \gamma_i \), we have

\[
\ell(\gamma_i) = \int_a^b \sqrt{E\dot{u}^2} \, dt \\
= \int_a^b \sqrt{2} \, dt \\
= (b - a)\sqrt{2}.
\]

Therefore, \( \ell(\gamma_1) = \ell(\gamma_2) \).
2. (25 points) A curve on a surface is called asymptotic if its normal curvature is everywhere zero.

(a) Determine the asymptotic curves of the surface $S$ defined by $z = xy$.

(b) Compute the Gaussian curvature of $S$.

Solution: We parametrize $S$ by $\sigma(u, v) = (u, v, uv)$. Thus

$$\sigma_u = (1, 0, v), \quad \sigma_v = (0, 1, u),$$

$$\sigma_u \times \sigma_v = \begin{vmatrix} i & j & k \\ 1 & 0 & v \\ 0 & 1 & u \end{vmatrix} = (-v, -u, 1),$$

so the corresponding unit normal is

$$N = \frac{\sigma_u \times \sigma_v}{\| \sigma_u \times \sigma_v \|} = \frac{(-v, -u, 1)}{\sqrt{1 + u^2 + v^2}}.$$

Let $\gamma(s) = (u(s), v(s), w(s))$ be a unit speed curve on $S$. Then $w(s) = u(s)v(s)$, so

$$w''(s) = u''(s)v(s) + 2u'(s)v'(s) + u(s)v''(s).$$

The normal curvature of $\gamma$ is therefore

$$\kappa_n = N \cdot \gamma'' = \frac{(-v, -u, 1)}{\sqrt{1 + u^2 + v^2}} \cdot (u'', v'', w'') = \frac{1}{\sqrt{1 + u^2 + v^2}}(-vu'' - uv'' + w'') = \frac{1}{\sqrt{1 + u^2 + v^2}}(2u'v').$$

Hence $\kappa_n = 0$ iff $u'(s)v'(s) = 0$, which is the case if either $u' = 0$ or $v' = 0$. Therefore, asymptotic curves of $S$ are the coordinate curves $u = \text{constant}$ and $v = \text{constant}$.

(b) First we observe that from (a) it follows that the coefficients of the first fundamental form are

$$E = 1 + v^2, \quad F = uv, \quad G = 1 + u^2.$$

Next we compute the coefficients of the second fundamental form of $\sigma$:

$$\sigma_{uu} = (0, 0, 0), \quad \sigma_{uv} = (0, 0, 1), \quad \sigma_{vv} = (0, 0, 0).$$
Therefore, \( L = N = 0 \) and

\[
M = N \cdot \sigma_{uv} = \frac{1}{\sqrt{1 + u^2 + v^2}}.
\]

The Gaussian curvature of \( S \) equals

\[
K = \frac{LN - M^2}{EG - F^2} = \frac{-\frac{1}{1 + u^2 + v^2}}{(1 + v^2)(1 + u^2) - u^2v^2} = -\frac{1}{(1 + u^2 + v^2)^2}.
\]
3. **(25 points)** Consider the parametrized surface

\[ \sigma(u, v) = \left( u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2 \right) \]

Show that:

(a) The coefficients of the first fundamental form are

\[ E = G = (1 + u^2 + v^2)^2, \quad F = 0. \]

(b) If \( L, M, N \) are the coefficients of the second fundamental form, then \( M = 0 \). (You don’t have to compute \( L \) and \( N \).)

(c) The lines of curvature are coordinate curves \( u = \text{constant} \) and \( v = \text{constant} \). Recall that a line of curvature of a surface is a curve \( \gamma \) such that \( \dot{\gamma}(t) \) is a principal vector for all \( t \).

**Solution:**

(a) We have

\[ \sigma_u = (1 - u^2 + v^2, 2uv, 2u), \quad \sigma_v = (2uv, 1 - v^2 + u^2, -2v), \]

so

\[ E = \|\sigma_u\|^2 = (1 - u^2 + v^2)^2 + 4u^2v^2 + 4u^2 \]

\[ = (1 + u^2 + v^2)^2, \]

\[ F = \sigma_u \cdot \sigma_v = 0, \quad G = \|\sigma_v\|^2 = E. \]

(b) To find \( M \), we need to compute the unit normal defined by \( \sigma \). We have

\[ \sigma_u \times \sigma_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 - u^2 + v^2 & 2uv & 2u \\ 2uv & 1 - v^2 + u^2 & -2v \end{vmatrix} \]

\[ = (-2u(1 + u^2 + v^2), 2v(1 + u^2 + v^2), 1 - (u^2 + v^2)^2). \]

Furthermore,

\[ \sigma_{uu} = (-2u, 2v, 2), \quad \sigma_{uv} = (2v, 2u, 0) \quad \sigma_{vv} = (2u, -2v, -2). \]

Since

\[ (\sigma_u \times \sigma_v) \cdot \sigma_{uv} = -2uv(1 + u^2 + v^2) + 2uv(1 + u^2 + v^2) = 0, \]

it follows that \( M = 0 \), as claimed.

(c) Let \( \mathbf{F}_I \) and \( \mathbf{F}_{II} \) be the matrices of the first and second fundamental forms of \( \sigma \). It follows from (a) and (b) that both of them are diagonal, as the off-diagonal entries \( F \) and \( M \) are zero. Hence the Weingarten matrix \( \mathbf{W} = \mathbf{F}_I^{-1}\mathbf{F}_{II} \) is diagonal. This implies that \((1, 0)^T\) and \((0, 1)^T\) are eigenvectors of \( \mathbf{W} \). Eigenvectors of \( \mathbf{W} \) are principal directions. Since vectors \((1, 0)^T\) and \((0, 1)^T\) are tangent to the curves \( u = \text{constant} \) and \( v = \text{constant} \), it follows that the coordinate curves are lines of curvature.
4. **(25 points)** Show that if a curve $\gamma$ on a surface $S$ is both a geodesic and an asymptotic curve, then $\gamma$ is a (segment of a) straight line.

**Solution:** We will use the classical formula

$$\kappa^2 = \kappa_g^2 + \kappa_n^2,$$

where $\kappa, \kappa_g$ and $\kappa_n$ denote the curvature, geodesic curvature and normal curvature, respectively. Suppose that $\gamma$ is a (unit speed) geodesic. Then $\kappa_g = 0$. If $\gamma$ is also an asymptotic curve, then $\kappa_n = 0$. The above formula implies $\kappa = 0$, hence $\dot{\gamma} = 0$. Therefore, $\gamma(s) = sa + b$, for some vectors $a, b$, so $\gamma$ is a segment of a straight line.