Midterm


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1. Denote by $m_d$ the Lebesgue measure on $\mathbb{R}^d$.

(a) If $A \subset \mathbb{R}$, $B \subset \mathbb{R}$, $m_1(A) = 0$ or $m_1(B) = 0$, show that $A \times B$ is measurable and $m_2(A \times B) = 0$.

(b) If $A, B \subset \mathbb{R}$ are open sets, show that $A \times B \subset \mathbb{R}^2$ is measurable and

$$m_2(A \times B) = m_1(A)m_1(B).$$

(c) (Extra credit) If $A, B \subset \mathbb{R}$ are bounded $G_\delta$-sets, show that $A \times B$ is measurable and

$$m_2(A \times B) = m_1(A)m_1(B).$$

Proof:
2. Let $A = C$ and $B = \frac{1}{2} C$, where $C$ be the Cantor middle third set. Show that 

$[0, 1] \subset A + B$. 

Therefore, there exist closed sets $A, B \subset \mathbb{R}$, with $m(A) = m(B) = 0$, but $m(A + B) > 0$. 

(As usual, $A + B = \{a + b : a \in A, b \in B\}$ and $\frac{1}{2} C = \{\frac{1}{2} c : c \in C\}$.)

Proof:
3. Let $f : E \to \mathbb{R}$ be a function, where $E \subset \mathbb{R}^d$ is a measurable set. Define a function 
$g : \mathbb{R}^d \to \mathbb{R}$ by

$$g(x) = \begin{cases} 
  f(x) & \text{if } x \in E \\
  0 & \text{if } x \notin E.
\end{cases}$$

Show that $f$ is measurable if and only if $g$ is measurable.

Proof:
4. A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be Borel measurable if for each $a \in \mathbb{R}$, the set $\{ x : f(x) > a \}$ is a Borel set.

(a) If $f$ is Borel measurable and $B$ is a Borel set, show that $f^{-1}(B)$ is a Borel set.

(b) If $f$ is Lebesgue measurable and $B$ is a Borel set, show that $f^{-1}(B)$ is a measurable set.

(c) If $f$ and $g$ are Borel measurable, show that so is $g \circ f$ (assuming the composition is defined).

(d) If $f$ is Lebesgue measurable and $g$ is Borel measurable, show that $g \circ f$ is Lebesgue measurable (assuming the composition is defined).

Proof: