1. Let $\mathcal{M}_{mn}(\mathbb{R})$ be the space of all real $m \times n$ matrices and $\mathcal{M}^k_{mn}(\mathbb{R})$ the subset of all of those $m \times n$ matrices whose rank is $\geq k$. Note that $\mathcal{M}_{mn}(\mathbb{R})$ can be identified with $\mathbb{R}^{mn}$ and is thus a smooth manifold.

Show that $\mathcal{M}^k_{mn}$ is an open subset of $\mathcal{M}_{mn}(\mathbb{R})$ and thus a manifold.

2. Let $\varphi_N$ be the stereographic projection from the north pole $(0, \ldots, 0, 1)$ of the unit sphere $S^n$ in $\mathbb{R}^{n+1}$, $\varphi_N : S^n \setminus \{N\} \to \mathbb{R}^n$. Denote the stereographic projection from the south pole $S = (0, \ldots, 0, -1)$ by $\varphi_S$.

(a) Compute $\varphi_N$ and $\varphi_S$.

(b) Show that $(U_N, \varphi_N)$ and $(U_S, \varphi_S)$, determine a smooth structure on $S^n$, where $U_N = S^n \setminus \{N\}$ and $U_S = S^n \setminus \{S\}$.

3. Compute the coordinate representation for each of the following maps in stereographic coordinates (see the previous problem) and use this to show that each map is smooth.

(a) For each $n \in \mathbb{Z}$, the $n^{th}$ power map $p_n : S^1 \to S^1$, given in complex notation by $p_n(z) = z^n$.

(b) The antipodal map $\alpha : S^n \to S^n$ defined by $\alpha(x) = -x$.

4. Suppose $F : M \to N$ is a smooth map between smooth manifolds, with $M$ connected, such that $T_pF : T_pM \to T_{F(p)}N$ is the zero map, for all $p \in M$. Show that $F$ is a constant map.

5. If a nonempty smooth $n$-manifold is diffeomorphic to an $m$-manifold, prove that $n = m$. 
